were made on sections at various distances from the fixed end of the beam. As extreme values they found $y_0=0.857d$ and $y_0=0.814d$. Formula (7) with $\beta=0.041$ and $\sigma=0.27$ gives $y_0=0.8414d$, a value which agrees well with the experimental results.

IOWA STATE COLLEGE OF AGRICULTURE AND MECHANIC ARTS

ON THE METRIZATION OF UNIFORM SPACE

L. W. COHEN AND CASPER GOFFMAN

We call a topological space S metrizable by a group G if there is a distance function for S with values in G which defines an equivalent metric topology in S. It should be noted that for the triangle inequality required of a distance to have meaning, the group G must be ordered, at least partially. In the classical metrization theorem of Chittenden [1],[1] G is the additive group of real numbers. The form of this theorem due to Weil [2] is that a necessary and sufficient condition for a uniform space to be metrizable by the group of real numbers is that it satisfy uniformly the first denumerability axiom of Hausdorff. Kalisch [3], using Weil's theorem that a uniform space is equivalent to a subset of a direct product of metric spaces with real distances, showed that every uniform space is metrizable by a partially ordered group. A related result due to Zelinsky [4] gives necessary and sufficient conditions that a topological field have a valuation in an ordered group.

Here we give necessary and sufficient conditions that a uniform space be metrizable by an ordered abelian group G. We shall assume that the identity in G is not isolated in the order of G, a restriction which excludes the topologically uninteresting discrete spaces. We have shown [5] that the topology of such a group G is determined by a limiting ordinal $\xi^*=\xi^*(G)$ with the property that if $\eta^*<\xi^*$ and ξ_η is a single-valued function on $\eta<\eta^*$ to $\xi<\xi^*$, then

$$(*)\qquad \sup\,[\xi_\eta \mid \eta < \eta^*] < \xi^*;$$

and elements a_ξ, $\xi<\xi^*$, decreasing in G, such that $\inf\,[a_\xi \mid \xi<\xi^*]=\theta$, the identity in G.

THEOREM. *A uniform space S is metrizable by an ordered abelian*

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

group G if and only if S has a neighborhood system $\mathcal{V}=[U_\xi(x),\ \xi<\xi^*,\ x\in S]$ *with the properties*

1. $\bigcap_{\xi<\xi^*}U_\xi(x)=[x]$,

2. $\xi_1<\xi_2<\xi^*$ *implies* $U_{\xi_1}(x)\supset U_{\xi_2}(x)$,

3. $\eta<\xi^*$ *implies there is a* $\xi(\eta)$ *such that* $\eta\leqq\xi(\eta)<\xi^*$ *and if* $U_{\xi(\eta)}(x)\cap U_{\xi(\eta)}(y)\neq 0$, *then* $U_{\xi(\eta)}(y)\subset U_\eta(x)$,

4′. $\eta^*<\xi^*$ *implies that* $\bigcap_{\eta<\eta^*}U_{\xi(\eta)}(x)$ *is open, where* ξ^* *is a limiting ordinal with the property* (*).

PROOF. As a preliminary, we remark that, since ω is the first limiting ordinal with the property (*), we may restrict ourselves to the case $\xi^*>\omega$ in view of the classical metrization theorem.

For the necessity: Suppose that $d(x, y)$ is a distance function for S with values in G. The metric topology in S is defined by the spheres $S(x,\ \xi)=[y\,|\,d(x,\ y)<a_\xi],\ \xi<\xi^*$. Property 1 holds since $\inf a_\xi=\theta$; property 2 holds since the a_ξ decrease on $\xi<\xi^*$; property 3 holds since $d(x, y)$ has the triangle property of a distance function; property 4′ holds since ξ^* is a limiting ordinal with property (*).

For the sufficiency: First we show that S has a neighborhood system $\mathcal{W}=[V_\xi(x)]$, equivalent to $\mathcal{V}$, with properties 1, 2, 3, 4′, and

5. $V_\xi(x)\cap v_\xi(y)\neq 0$ *implies* $V_\xi(x)=V_\xi(y)$.

For $\eta<\xi^*$ we put $\eta_0=\eta$ and $\eta_n=\xi(\eta_{n-1}),\ n<\omega$, where $\xi(\eta_{n-1})$ is the ordinal of 3. Since $\omega<\xi^*$, $\lambda(\eta)=\sup\,[\eta_n\,|\,n<\omega]<\xi^*$. We define $\zeta(\eta)=\sup\,[\lambda(\mu)\,|\,\mu<\eta+1]$. From 3 and (*) we have $\eta\leqq\xi(\eta)\leqq\lambda(\eta)\leqq\zeta(\eta)<\xi^*$ and $\zeta(\eta_1)\leqq\zeta(\eta_2)$ if $\eta_1<\eta_2$. We define $V_\eta(x)$ as the set of y for which there are $n<\omega$ and $x_1=x,\ x_2,\ \cdots,\ x_n=y\in S$ such that $U_{\zeta(\eta)}(x_k)\cap U_{\zeta(\eta)}(x_{k+1})\neq 0$ for $k=1,\ \cdots,\ n-1$.

The equivalence of $\mathcal{W}=[V_\eta(x)]$ and $\mathcal{V}$ is established by showing that $U_{\zeta(\eta)}(x)\subset V_\eta(x)\subset U_\eta(x),\ \eta<\xi^*$. The first inclusion follows from $y\in U_{\zeta(\eta)}(x)$ and $x_1=x,\ x_2=y$. For the second, consider $y\in V_\eta(x)$. Then there are $n<\omega$ and $x_1=x,\ x_2,\ \cdots,\ x_n=y\in S$ such that $U_{\zeta(\eta)}(x_k)\cap U_{\zeta(\eta)}(x_{k+1})\neq 0,\ k=1,\ \cdots,\ n-1$. From 2 and $\eta_k\leqq\lambda(\eta)\leqq\zeta(\eta)$, we have $U_{\eta_k}(x_k)\cap U_{\eta_k}(x_{k+1})\neq 0$. Since $\eta_k=\xi(\eta_{k-1})$ we have, from 3, $U_{\eta_k}(x_{k+1})\subset U_{\eta_{k-1}}(x_k)$. Hence

$$y = x_n \in U_{\eta_{n-1}}(x_n) \subset U_{\eta_{n-2}}(x_{n-1}) \subset \cdots \subset U_{\eta_0}(x_1) = U_\eta(x)$$

and so $V_\eta(x)\subset U_\eta(x)$.

That the system $\mathcal{W}$ has properties 1 and 4′ follows from its equivalence with $\mathcal{V}$. For property 2 consider $\eta'<\eta''$. If $y\in V_{\eta''}(x)$, then for some $x_1=x,\ x_2,\ \cdots,\ x_n=y,\ U_{\zeta(\eta'')}(x_k)\cap U_{\zeta(\eta'')}(x_{k+1})\neq 0,\ k=1,\ \cdots,\ n-1$. Since $\mathcal{V}$ has property 2 and $\zeta(\eta')\leqq\zeta(\eta'')$, $U_{\zeta(\eta')}(x_k)\cap U_{\zeta(\eta')}(x_{k+1})\neq 0$. Hence $y\in V_{\eta'}(x)$ and $\mathcal{W}$ has property 2. For property 3 consider

$\eta < \xi^*$ and the ordinal $\xi(\zeta(\eta))$ corresponding to $\zeta(\eta)$ by property 3 for the system $\mathcal{V}$. Now by the inclusions of the equivalence argument, $V_{\xi(\zeta(\eta))}(y) \cap V_{\xi(\zeta(\eta))}(x) \neq 0$ implies $U_{\xi(\zeta(\eta))}(y) \cap U_{\xi(\zeta(\eta))}(x) \neq 0$ and so $V_{\xi(\zeta(\eta))}(y) \subset U_{\xi(\zeta(\eta))}(y) \subset U_{\zeta(\eta)}(x) \subset V_\eta(x)$. Since $\eta \leqq \zeta(\eta) \leqq \xi(\zeta(\eta))$, $\xi(\zeta(\eta))$ serves as the ordinal corresponding to η in property 3 for $\mathcal{W}$.

For property 5 consider $w \in V_\eta(x) \cap V_\eta(y)$ and $z \in V_\eta(y)$. Then there are $x_1' = x, x_2', \cdots, x_p' = w; y_1' = y, y_2', \cdots, y_q' = w; y_1'' = y, y_2'', \cdots, y_r'' = z$ in S such that $U_{\zeta(\eta)}(u_k) \cap U_{\zeta(\eta)}(u_{k+1}) \neq 0$ if $u_k = x_k'$, $k = 1, \cdots, p-1$; $u_k = y_k'$, $k = 1, \cdots, q-1$; $u_k = y_k''$, $k = 1, \cdots, r-1$, respectively. We now have $x_1 = x$, $x_2, \cdots, x_{p+q+r-2} = z$ and $U_{\zeta(\eta)}(x_k) \cap U_{\zeta(\eta)}(x_{k+1}) \neq 0$, $k = 1, \cdots, p+q+r-3$. Hence $z \in V_\eta(x)$ and 5 follows easily.

The ordered group G by which S is metrizable is the additive group of real functions on the ordinals $\xi < \xi^*$. For: Given $x, y \in S$ and $\xi < \xi^*$ we define

$$d_\xi(x, y) = \begin{cases} 0, & V_\xi(x) = V_\xi(y), \\ 1, & V_\xi(x) \neq V_\xi(y). \end{cases}$$

For each pair $x, y \in S$ the $d_\xi(x, y)$ are the values of a real function $d(x, y)$ on $\xi < \xi^*$. It is clear that $d(x, y) = d(y, x) \geqq \theta$, the identity in G. Since $\mathcal{W}$ has property 1, $d(x, y) = \theta$ if and only if $x = y$. For the triangle property of $d(x, y)$, consider $x, y, z \in S$ and $\xi < \xi^*$ such that $d_\xi(x, z) = d_\xi(z, y) = 0$. Then $V_\xi(x) = V_\xi(z) = V_\xi(y)$ and so $d_\xi(x, y) \leqq d_\xi(x, z) + d_\xi(z, y)$.

Now by the definition of order in G we have $d(x, y) \leqq d(x, z) + d(z, y)$ for all $x, y, z \in S$. Hence $d(x, y)$ is a distance function for S with values in G.

Finally we show that S is metrizable by the group G. Let $a_\xi \in G$, $\xi < \xi^*$, be the functions

$$a_\xi(\eta) = \begin{cases} 1, & \eta = \xi, \\ 0, & \eta \neq \xi. \end{cases}$$

For each $\xi < \xi^*$ and $x \in S$, $V_\xi(x) = [y \mid d(x, y) < a_\xi] = S(x, \xi)$. For: Consider $y \in V_\xi(x)$. By property 2, $y \in V_\eta(x)$, $\eta \leqq \xi$, and so, by property 5, $d_\eta(x, y) = 0$ for $\eta \leqq \xi$. Hence $d(x, y) < a_\xi$. Conversely, if $d(x, y) < a_\xi$, $d_\eta(x, y) = 0$ for $\eta \leqq \xi$ and so $V_\xi(x) = V_\xi(y)$. From property 1 it follows that $y \in V_\xi(x)$.

We conclude with 3 remarks: 1. If $\xi^* > \omega$, the metric for S is non-archimedean. 2. The spaces considered in [6] are metrizable by ordered abelian groups. 3. The space of real functions of a real variable with the weak topology is not metrizable by an ordered abelian group.

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Queens College and
University of Oklahoma