CONCERNING THE GENERATORS OF HOMOTOPY GROUPS OF A POLYHEDRON

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1. One of the outstanding problems in homotopy theory is to determine the structure of homotopy groups of a given topological space. Even for many simple spaces, very little is known. Given a finite polyhedron P, we also do not know in general whether or not $\pi_n(P)$ is generated by a finite number of elements.

In this note, the following theorem will be proved.

THEOREM 1. *Let P be a finite connected n-dimensional polyhedron with $n \geqq 2$. Let $\pi_t(P)=0$ for all $1<t<n$ but $\pi_n(P) \neq 0$. Then, $\pi_n(P)$ is a group with a finite number of generators if and only if $\pi_1(P)$ is a finite group.*

Let us collect before the proof some notations and elementary facts to be used later. Given a simplicial complex J, by the notations $H_n(J)$ and $Z_n(J)$ will be meant respectively the nth homology group and the group of n-cycles of J, formed by using finite chains with integral coefficients.

Let P be a connected polyhedron and S be the n-sphere $\sum_{i=0}^{n} x_i^2 =1$ in Euclidean $(n+1)$-space $(n \geqq 2)$. Let ϕ be a mapping: $S \to P$. Then, given any $s, s' \in S$, a path $\lambda_{ss'}$ in S from s to s' gives rise to a path $\phi(\lambda_{ss'})$ in P from $\phi(s)$ to $\phi(s')$.[1] We shall write $\phi_{ss'}$ for $\phi(\lambda_{ss'})$. $\phi_{ss'}$ is determined uniquely by s and s', since S is simply connected.[1] Clearly, $\phi_{ss'} = \phi_{s's}^{-1}$ and $\phi_{ss'} \cdot \phi_{s's''} = \phi_{ss''}$.

Now, let L be a simplicial decomposition of P and p^* be a vertex of L. As usual,[2] we are able to construct a simplicial complex $\tilde{L}$ subject to the following conditions:

(i) There is a one-to-one transformation f from the set of vertices of $\tilde{L}$ to the set of all ρ_q's where q is a vertex of L, and ρ_q is a path in P from p^* to q.

(ii) Let g be the transformation which carries ρ_q to q and let $\theta = gf$. Then, any $k+1$ mutually distinct vertices $\tilde{q}_0, \tilde{q}_1, \cdots, \tilde{q}_k$ of $\tilde{L}$ span a k-simplex of $\tilde{L}$ if and only if $\theta(\tilde{q}_0), \theta(\tilde{q}_1), \cdots, \theta(\tilde{q}_k)$ are the distinct vertices of a k-simplex of L and $f(\tilde{q}_{j'})$ is the resultant of $f(\tilde{q}_j)$ multiplied by the path represented by the oriented segment from $\theta(\tilde{q}_j)$ to $\theta(\tilde{q}_{j'})$.

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[1] $\lambda_{ss'}$ is a homotopy class rel. 0 and 1 of mappings g: $\langle 0, 1\rangle \to S$ with $g(0)=s$ and $g(1)=s'$. $\phi(\lambda_{ss'})$ is the homotopy class rel. 0 and 1 in P containing the mappings ϕg.

Denote by $\tilde{P}$ the polyhedron built of the simplexes of $\tilde{L}$. Then, θ gives rise naturally to a simplicial mapping θ: $\tilde{P}\rightarrow P$. It is known[2] that $\tilde{P}$ is a universal covering space of P with covering mapping θ. We have

$$\pi_1(\tilde{P}) = 0. \tag{1}$$

Let us assume further that

$$\pi_t(P) = 0 \text{ for a certain } n \geqq 2 \text{ and all } 1 < t < n. \tag{2}$$

Then, by the covering mapping θ,

$$\pi_t(\tilde{P}) = 0 \text{ for all } 1 < t < n, \qquad \pi_n(\tilde{P}) \approx \pi_n(P). \tag{3}$$

This together with (1) and an isomorphism of Hurewicz[3] gives

$$\pi_n(P) \approx H_n(\tilde{L}). \tag{4}$$

Thus, under the condition (2), we have: (i) If P is a finite polyhedron and $\pi_1(P)$ is a finite group, then $\tilde{L}$ is a finite complex and $H_n(\tilde{L})$ as well as $\pi_n(P)$ is a group with a finite number of generators; (ii) if dim $P=n$, then dim $\tilde{P}=n$ and $H_n(\tilde{L})\approx Z_n(\tilde{L})$ and hence it follows from (4) that

$$\pi_n(P) \approx Z_n(\tilde{L}). \tag{5}$$

Proof of Theorem 1. The proof of the sufficiency of the theorem has been given above. The necessity follows from the following more general statement (Δ).

Let P be a connected n-dimensional polyhedron (not necessarily finite) which fulfils the condition (2). *If $\pi_1(P)$ is an infinite group and $\pi_n(P)$ $\neq 0$, then $\pi_n(P)$ contains an infinite number of independent elements.*[4]

We now prove the statement (Δ). First, taking a vertex p^* of a simplicial decomposition L of P as base point, construct the universal covering space $\tilde{P}$ of P as above. Let the notations $\tilde{L}, f, \theta, S$ retain their usage here. Let s^* be a point of S and let η be a nonzero element of $\pi_n(P, p^*)$. Decompose S into a simplicial complex K so that we have

[2] H. Seifert and W. Threfall, *Lehrbuch der Topologie*, Leipzig, Teubner, 1934, pp. 189–194.

[3] A mapping h: $S\rightarrow P$, where S is an oriented n-sphere, and P an arcwise connected topological space, determines a singular cycle $\psi(h)=(h, S)$ in P. If $\pi_t(P)=0$ for all $0<t<n$, ψ gives rise to an isomorphism of $\pi_n(P)$ onto the nth singular homology group of P with integral coefficients. This isomorphism will be referred to as an isomorphism of Hurewicz.

[4] We mean that $\pi_n(P)$ contains an infinite set A such that, for each positive integer k, k mutually distinct elements of A are independent.

a simplicial mapping ϕ: $(S, s^*)\to(P, p^*)$ which represents η. Denote by B the set of all elements of π_1 (P, p^*) of the form $\phi_{s^*r}\cdot\phi_{r's^*}$ where r and r' are vertices of K with $\phi(r)=\phi(r')$. Clearly, B is a finite set.

Suppose that $\pi_1(P, p^*)$ consists of an infinite number of elements. Let us define a sequence $\{\xi_i\}$ of elements of $\pi_1(P, p^*)$, taking ξ_1 arbitrarily first, and then, by mathematical induction on supposing that $\xi_1, \xi_2, \cdots, \xi_i$ have already been defined, taking ξ_{i+1} as an arbitrary element of $\pi_1(P, p^*)$ which does not belong to the finite set $\bigcup_{j=1}^{i} \xi_j\cdot B$.[5] We see that the sequence $\{\xi_i\}$ so obtained possesses the property that, for any $i\neq i'$, $\xi_i^{-1}\cdot\xi_{i'}\notin B$.

To each ξ_i, we are able to associate a transformation ψ_i from the vertices of K to the vertices of $\tilde{L}$ subject to the condition:

$$f\psi_i(r) = \xi_i\cdot\phi_{s^*r} \text{ for any vertex } r \text{ of } K. \tag{6}$$

Since $\xi_i\cdot\phi_{s^*r}(\xi_i\cdot\phi_{s^*r})\cdot\phi_{rr'}$ for any vertices r and r' of K, by the construction of $\tilde{P}$, ψ_i gives rise naturally to a simplicial mapping ψ_i: $S\to\tilde{P}$. Clearly $\phi=\theta\psi_i$. Let us denote by α the fundamental n-cycle of K. Then, $\psi_i(\alpha)\in Z_n(\tilde{L})$.

We see that every $\psi_i(\alpha)\neq 0$. For, if $\psi_i(\alpha)=0$, then, since P fulfills (2), we have from (1), (3), and an isomorphism of Hurewicz[3] that $\psi_i\cong 0$ in $\tilde{P}$; it follows that $\phi\simeq 0$ in P, which contradicts that $\eta\neq 0$. We see also that for any two distinct i and i', the absolute complexes $|\psi_i(\alpha)|$ and $|\psi_{i'}(\alpha)|\subset\tilde{L}$ are disjoint. For, if not, there are vertices r and r' of K such that $\psi_i(r)=\psi_{i'}(r')$ and hence by (6) $\xi_i\cdot\phi_{s^*r}=\xi_{i'}\cdot\phi_{s^*r'}$ which contradicts that $\xi_{i'}^{-1}\cdot\xi_i\notin B$. Moreover, the cycles $\psi_i(\alpha)$'s are infinite in number. It follows therefore that $Z_n(\tilde{L})$ contains an infinite number of independent elements.

Since P fullfils the condition (2) and dim $P=n$, we have therefore the isomorphism (5). This completes the proof of the statement (Δ) and hence that of Theorem 1.

2. Let n be an integer not less than 2. Given any two pathwise connected topological spaces $P\subset P'$ and p^*, a point of P, we shall always use ι to denote the injection homomorphism: $\pi_n(P, p^*)\to\pi_n(P', p^*)$. Now, let us consider three pathwise connected topological spaces Q, $R\subset P$. Let s^* be a point of $Q\cap R$, and let ι^* be the homomorphism: $\pi_n(Q, s^*)+\pi_n(R, s^*)$ [direct sum]$\to\pi_n(P, s^*)$ defined by taking $\iota^*(\eta)=\iota(\xi)+\iota(\zeta)$ where $\xi\in\pi_n(Q, s^*)$, $\zeta\in\pi_n(R, s^*)$ and $\eta=\xi+\zeta$. It follows from some arguments given by G. W. Whitehead that:[6]

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[5] $\xi_i\cdot B$ is the set of all such element $\xi_i\cdot\zeta$ of $\pi_1(P, p^*)$ with $\zeta\in B$.

[6] G. W. Whitehead, *A generalization of the Hopf invariant*, Proc. Nat. Acad. Sci. U.S.A. vol. 32 (1946) pp. 188–190.

LEMMA 1. *If $Q \cup R = P$ and $Q \cap R = (s^*)$, ι^* is an into-isomorphism.*

THEOREM 2. *Let P be a connected polyhedron, and P', the polyhedron obtained by identifying two distinct points p_1 and p_2 of P to a single point. Then, if $\pi_n(P) \neq 0$ for $n \geqq 2$, $\pi_n(P')$ cannot be a group with a finite number of generators.*

To prove the theorem, we construct a polyhedron

$$Q = \bigcup_{i=0,\pm 1,\pm 2,\cdots} Q_i$$

subject to the conditions: (i) For each i, there is a homeomorphism f_i of Q_i onto P; (ii) Q_i and Q_{i+1} are in contact at the point $q_i = f_i^{-1}(p_1) = f_{i+1}^{-1}(p_2)$ and have no other common points than q_i; (iii) $Q_i \cap Q_{i+j} = 0$ if $j > 1$. Clearly, Q is a covering space of P'. We have

$$\pi_n(Q_i) \approx \pi_n(P) \text{ for every } i, \tag{7}$$

and since $n \geqq 2$, we have

$$\pi_n(Q) \approx \pi_n(P'). \tag{8}$$

Given any pair of integers $m \leqq m'$, denote by $Q_{mm'}$ the subpolyhedron $\bigcup_{m \leqq i \leqq m'} Q_i$ of Q. Making use of Lemma 1 by changing successively the base point to form a homotopy group, we shall have a subgroup of $\pi_n(Q_{mm'})$ which is isomorphic to the direct sum $\pi_n(Q_m) + \pi_n(Q_{m+1}) + \cdots + \pi_n(Q_{m'})$ and which the injection homomorphism $\iota: \pi_n(Q_{mm'}) \to \pi_n(Q)$ maps isomorphically into $\pi_n(Q)$. Therefore, by (7), given any positive integer k, $\pi_n(Q)$ contains a subgroup isomorphic to the direct sum of k groups $\approx \pi_n(P)$. This together with (8) and some arguments in group theory proves Theorem 2.

COROLLARY. *Under the same hypotheses of Theorem 2, if $n \geqq 2$ and $\pi_n(P') \neq 0$, then $\pi_n(P')$ cannot be a group with a finite number of generators.*

To prove this, as in the proof of Theorem 2, we construct the covering space Q of P'. If $n \geqq 2$ and $\pi_n(P') \neq 0$, then we have an essential mapping of the n-sphere into some $Q_{mm'}$, that is, $\pi_n(Q_{mm'}) \neq 0$ for some $Q_{mm'}$. Clearly, Q is also a covering space of a certain polyhedron $Q'_{mm'}$ obtained by identifying two distinct points of $Q_{mm'}$. It follows therefore from Theorem 2 that $\pi_n(P') \approx \pi_n(Q) \approx \pi_n(Q'_{mm'})$ is not a group with a finite number of generators.

THEOREM 3. *Let P be a connected polyhedron, and P' the polyhedron obtained by identifying two distinct points p_1 and p_2 of P. Then, if $n \geqq 2$ and $\pi_t(P) = 0$ for all $1 < t < n$, $\pi_t(P')$ is the weak direct sum of an infinite countable number of groups each of which is isomorphic to $\pi_n(P)$.*

We precede the proof of the theorem by two lemmas. Given two polyhedra $P \subset P^*$, and two points $p_1 \neq p_2$ of P, we shall say that P^* is a segmental extent of P through the points p_1 and p_2, if there is a simplicial decomposition L^* of P^* such that p_1 and p_2 span a ground 1-simplex of L^* and P is built of all simplexes of L^* other than the simplex $|p_1p_2|$, that is, $P = \overline{L^* - |p_1p_2|}$.

Lemma 2. *Let P^* be a segmental extent of a polyhedron P through the points $p_1 \neq p_2$ of P, and let P' be the polyhedron obtained by identifying the points p_1 and p_2. Then P^* and P' have the same homotopy type.*

This can be easily obtained by some standard procedure in homotopy theory. The detailed proof will be omitted here.

Lemma 3. *Let P^* be a segmental extent of a connected polyhedron P through the points $p_1 \neq p_2$ of P. Then*: (i) *The injection homomorphism $\iota: \pi_1(P, p_1) \rightarrow \pi_1(P^*, p_1)$ is an into-isomorphism*; (ii) *$\pi_1(P^*, p_1)$ is isomorphic to the free product of $\pi_1(P, p_1)$ and an infinite cyclic group.*

To prove this, by the definition of segmental extents, there is a simplicial decomposition L^* of P^* such that p_1 and p_2 span a ground 1-simplex of L^* and $P = \overline{L^* - |p_1p_2|}$. Since P is connected, there is a simple arc A contained in the 1-skeleton of P and joining p_1 and p_2. Clearly, $S^1 = |p_1p_2| \cup A$ is homeomorphic to a circle.

Denote also by ι the injection homomorphism: $\pi_1(S^1, p) \rightarrow \pi_1(P^*, p_1)$. Given an arbitrary element $\xi = \xi_1\xi_2 \cdots \xi_t$ of the free product $F = \pi_1(P, p_1) \circ \pi_1(S^1, p_2)$ where $\xi_i \in \pi_1(P, p_1)$ or $\pi_1(S^1, p_1)$, put $\iota^*(\xi) = \iota(\xi_1)\iota(\xi_2) \cdots \iota(\xi_t)$. Then, since $\pi_1(A, p_1) = 1$, ι^* establishes an onto-isomorphism: $F \rightarrow \pi_1(P^*, p_1)$.[7] (ii) follows now therefore at once, since $\pi_1(S^1, p_1)$ is cyclic infinite. (i) follows from the fact that the homomorphism $\iota: \pi_1(P, p_1) \rightarrow \pi_1(P^*, p_1)$ agrees with ι^* over the subgroup $\pi_1(P, p_1)$ of F.

Proof of Theorem 3. Without loss of generality, we may assume that P is a polyhedron contained in a sufficiently higher-dimensional Euclidean space or in the Hilbert space so that it has a segmental extent P^* through p_1 and p_2. Let L^* be a simplicial decomposition of P^* such that p_1 and p_2 span a ground 1-simplex of L^* and $P = \overline{L^* - |p_1p_2|}$. By Lemma 2 it suffices for us to prove that $\pi_n(P^*)$ is the weak direct sum of an infinite countable number of groups $\approx \pi_n(P)$. Let us construct as in §1 a simplicial complex $\tilde{L}^*$ such that the polyhedron $\tilde{P}^*$ built of the simplexes of $\tilde{L}^*$ is a universal covering space of P^* with a simplicial covering mapping θ.

Denote by $\tilde{P}_i$ an arbitrary component of $\theta^{-1}(P)$. $\tilde{P}_i$ is then a cover-

[7] H. Seifert and W. Threfall, loc. cit. pp. 177–179.

ing space of P with covering mapping $\theta | \tilde{P}_i$. Let f be a mapping: $S^1 \rightarrow \tilde{P}_i$ where S^1 is a circle. Then $f \simeq 0$ in $\tilde{P}^*$ and hence $\theta f \simeq 0$ in P^*. It follows from Lemma 3(i) that $\theta f \simeq 0$ in P and hence by the covering mapping $\theta | \tilde{P}_i$ that $f \simeq 0$ in $\tilde{P}_i$. Thus, $\tilde{P}_i$ is also a universal covering space of P. Next, we see that every $\tilde{P}_i$ is a polyhedron composed of the simplexes of a subcomplex $\tilde{L}_i$ of $\tilde{L}^*$. Clearly, a simplex of $\tilde{L}^*$ is of dimension 1 if it is not in any $\tilde{L}_i$. We have therefore that $H_t(\tilde{L}^*)$ is the weak direct sum of the $H_t(\tilde{L}_i)$'s for $t \geqq 2$. By a theorem of Hurewicz,[3] we obtain easily that $\pi_n(P^*)$ is the weak direct sum of the $\pi_n(\tilde{P}_i)$'s.

Since $\pi_n(\tilde{P}_i) \approx \pi_n(P)$, it remains only to show that the $\tilde{P}_i$'s are countably infinite. But this follows from the fact that the components of $\theta^{-1}(P)$ are in one-one correspondence with the collection of left cosets of $i(\pi_1(P))$ in $\pi_1(P^*)$. By Lemma 3 this collection is countably infinite.

Repeating the use of Theorem 3 by taking $n = 2, 3, 4, \cdots$, successively, we get the following corollary.

COROLLARY. *Let P be a connected polyhedron, and P' the polyhedron obtained by identifying two distinct points of P. Then P' is aspherical if and only if P is aspherical.*

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