CONVEXICAL BLANKETS

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1. **Introduction.** Suppose given a metric space $\mathfrak{S}$, a set $A \subset \mathfrak{S}$, and a function which correlates with each point z in A such a family of nonvacuous subsets of $\mathfrak{S}$ that there are members of this family of diameter arbitrarily small, shrinking down upon, but not necessarily containing, the point z. Such a function is called a *blanket*[1] with domain A. It is known that if a blanket possesses certain covering properties, then almost everywhere differentiation is possible with respect to it.[2]

Herein we restrict ourselves to the study of a particular kind of blanket in the euclidean space of n dimensions $\mathfrak{R}_n$. A blanket F of the type under consideration is so set up that for z in its domain, $F(z)$ is defined with the aid of a certain family of convex sets which we call a *generator*. It will be shown in §3 that if ϕ is a measure in $\mathfrak{R}_n$ of a rather arbitrary nature, then we can always determine a *subblanket* of F which is *ϕ-regular* and which may be used for purposes of differentiation. This is true even when the convex sets in the generator are arbitrarily thin. Results of a stronger nature are known[3] if ϕ is taken to be Lebesgue measure in $\mathfrak{R}_n$. In §4, we devote our attention to the construction of an example which serves to indicate certain limitations upon the results which one might hope to obtain.

2. **Preliminaries.** In this section we state a number of definitions and conventions which are needed subsequently. It proves convenient to keep our terminology and notation generally in accord with M.

Let us agree that $z \in A = (z$ is a member of $A) = (z$ is in $A)$; $(z \notin A) = (z$ is not a member of $A) = (z$ is not in $A)$; $(A \subset B) = (A$ is a subset of $B) = (B \supset A) = (B$ contains $A)$; the integer 0 and the null set are identical.

If a_t is a set for each $t \in B$, then

$$\sum_{t \in B} a_t = E_z[(z \in a_t) \text{ for some } t \in B].$$

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[1] Unfamiliar technical terms are defined in §2.

[2] See A. P. Morse, *A theory of covering and differentiation*, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 205–235. Hereinafter this will be referred to as M.

[3] See M, pp. 215–216.

If $\mathfrak{F}$ is a family, then

$$\sigma\mathfrak{F} = \sum_{\beta \in \mathfrak{F}} \beta.$$

Thus $t \in \sigma\mathfrak{F}$ if and only if t is a member of some set in $\mathfrak{F}$.

If $a_t \geqq 0$ for each t in a countable set B, then

$$\sum_{t \in B} a_t$$

also denotes the appropriate numerical sum. However, it will be clear from the context whether set theoretic or numerical summation is intended in a particular instance.

Throughout §2 and §3 we let n denote a fixed positive integer, and we denote by $\mathfrak{R}_n$ euclidean space of n dimensions. We let ρ be that function for which

$$\rho(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$$

whenever $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ are points in $\mathfrak{R}_n$; ρ is the metric of $\mathfrak{R}_n$. Henceforth such terms as distance, interior, closed, Borel, and so on, will refer to the metric ρ.

If $\beta \subset \mathfrak{R}_n$, then K_β will denote the characteristic function of β.

If r is a real number, x and y are points in $\mathfrak{R}_n$, then rx, $x+y$, and $x-y$ will be understood to have their usual vectorial meanings.

2.1. DEFINITION.

$$C(z, r) = \underset{t}{E} \left[\rho(t, z) \leqq r \right].$$

Clearly $C(z, r)$ is the closed sphere of center z and radius r whenever $z \in \mathfrak{R}_n$ and $r \geqq 0$.

2.2. DEFINITION. For $A \subset \mathfrak{R}_n$ we define

$$\operatorname{diam} A = \sup_{(x \in A)(y \in A)} \rho(x, y) \qquad \text{if } A \neq 0;$$

$$\operatorname{diam} A = 0 \qquad \text{if } A = 0.$$

2.3. DEFINITION. ϕ *measures* $\mathfrak{R}_n$ if and only if ϕ is such a function on

$$\underset{\beta}{E} \left[\beta \subset \mathfrak{R}_n \right] \quad \text{to} \quad \underset{t}{E} \left[0 \leqq t \leqq \infty \right]$$

that

$$\phi(A) = \sum_{\beta \in \mathfrak{F}} \phi(\beta)$$

whenever $\mathfrak{F}$ is a countable family for which $A \subset \sigma\mathfrak{F} \subset \mathfrak{R}_n$.

2.4. DEFINITION. The family $\mathfrak{U}$ is defined by: $\phi \in \mathfrak{U}$ if and only if ϕ so measures $\mathfrak{R}_n$ that

$$\phi(A + B) = \phi(A) + \phi(B)$$

whenever

$$\inf_{(x \in A)(y \in B)} \rho(x, y) > 0.$$

2.5. DEFINITION.

$$\mathfrak{B} = \mathfrak{U} \cdot \underset{\phi}{E} [\phi(\beta) < \infty \text{ whenever } \beta \text{ is a bounded subset of } \mathfrak{R}_n].$$

2.6. DEFINITION. A is ϕ-measurable if and only if ϕ so measures $\mathfrak{R}_n$ that

$$\phi(T) = \phi(AT) + \phi(T - A)$$

for each subset T of $\mathfrak{R}_n$. It is well known that all Borel sets are ϕ-measurable whenever $\phi \in \mathfrak{B}$.

2.7. DEFINITION.

$$\text{sng } z = \underset{t}{E} [t = z].$$

Thus sng z is the set whose sole member is z.

2.8. DEFINITION. the limit notations

$$\underset{\mathfrak{F} \ni \beta \to z}{\text{limalinf}} f(\beta), \qquad \underset{\mathfrak{F} \ni \beta \to z}{\text{limalsup}} f(\beta)$$

are defined, respectively, as

$$\lim_{t \to 0+} \left(\inf_{\beta \in H(t)} f(\beta) \right), \qquad \lim_{t \to 0+} \left(\sup_{\beta \in H(t)} f(\beta) \right),$$

where

$$H(t) = \mathfrak{F} \cdot \underset{\beta}{E} [\text{diam } (\beta + \text{sng } z) < t].$$

We write

$$\underset{\mathfrak{F} \ni \beta \to z}{\text{limal}} f(\beta) = \underset{\mathfrak{F} \ni \beta \to z}{\text{limalsup}} f(\beta)$$

if and only if

$$\underset{\mathfrak{F} \ni \beta \to z}{\text{limalsup}} f(\beta) = \underset{\mathfrak{F} \ni \beta \to z}{\text{limalinf}} f(\beta).$$

2.9. DEFINITION. We say F is a *blanket* if and only if F is such a function that for z in its domain

(i) $z \in \mathfrak{R}_n$ and $F(z)$ is a family of nonvacuous subsets of $\mathfrak{R}_n$;
(ii) $\beta \in F(z)$ implies $(\operatorname{diam} \beta) < \infty$;
(iii)
$$\inf_{\beta \in F(z)} \{\operatorname{diam} (\beta + \operatorname{sng} z)\} = 0.$$

2.10. DEFINITION. If F is a blanket with domain A then the set
$$\sum_{z \in A} F(z)$$
is called the *spread* of F.

2.11. DEFINITION. F is a *natural* blanket if and only if F is a blanket and $z \in \beta$ whenever $\beta \in F(z)$.

2.12. DEFINITION. F is a *Borelian* (*close*) blanket if and only if F is such a blanket that β is a Borel (closed) set whenever β is in the spread of F.

2.13. DEFINITION. If Δ is a non-negative function whose domain is a family of sets H, then corresponding to each β in H we define $\Delta:\beta$ as the set S for which $z \in S$ if and only if there is such a set α in H that
$$z \in \alpha, \quad \alpha\beta \neq 0, \quad \text{and} \quad \Delta(\alpha) \leqq 2\Delta(\beta).$$

2.14. DEFINITION. If F is a natural blanket, $\phi \in \mathfrak{B}$, and there is such a non-negative function Δ whose domain is the spread of F that
$$\operatorname*{limalsup}_{F(z) \ni \beta \to z} \left[\Delta(\beta) + \frac{\phi(\Delta:\beta)}{\phi(\beta)} \right] < \infty$$
for ϕ-almost all z in the domain of F, then F is said to be a *ϕ-regular* blanket.

2.15. DEFINITION. The family $\mathfrak{F}$ *covers* A if and only if $A \subset \sigma\mathfrak{F}$.

2.16. DEFINITION. The family $\mathfrak{F}$ *covers ϕ-almost all of A* if and only if $\phi \in \mathfrak{B}$ and $\phi(A - \sigma\mathfrak{F}) = 0$.

2.17. DEFINITION. If $\phi \in \mathfrak{B}$, F is a Borelian blanket with domain A, and the spread of F contains a countable disjointed family which covers ϕ-almost all of A, then F is a *ϕ-heavy* blanket.

2.18. DEFINITION. G is a *subblanket* of F if and only if F and G are such blankets that $G(z) \subset F(z)$ whenever z is in the domain of G.

2.19. DEFINITION. If F is such a blanket that each of its subblankets is ϕ-heavy, then F is a *ϕ-strong* blanket.

The following is proved in M.

2.20. THEOREM. *Every ϕ-regular close blanket is ϕ-strong.*

2.21. DEFINITION. A family $\mathfrak{H}$ is a *nest* if and only if ($\beta_1 \in \mathfrak{H}$ and $\beta_2 \in \mathfrak{H}$) implies ($\beta_1 \subset \beta_2$ or $\beta_2 \subset \beta_1$).

2.22. DEFINITION. If $\mathfrak{H}$ is any family of subsets of $\mathfrak{R}_n$, and $z \in \mathfrak{R}_n$, then the *z-translate* of $\mathfrak{H}$ is that family of sets obtained by translating each member of $\mathfrak{H}$ by the vector z; it is denoted by $\mathfrak{H}_z$.

2.23. DEFINITION. $\mathfrak{H}$ is said to be a *generator* if and only if $\mathfrak{H}$ is such a nonvacuous nest that

(i) $\beta\in\mathfrak{H}$ implies that β is a bounded, closed, symmetrical convex set with interior points, whose centroid is at the origin;

$$\text{(ii)} \qquad \inf_{\beta\in\mathfrak{H}} (\operatorname{diam} \beta) = 0.$$

2.24. DEFINITION. For any set $\beta\subset\mathfrak{R}_n$ with centroid a, and any $r\geqq 0$, we agree that β^r is the set of points of the form $a+r(x-a)$, where $x\in\beta$.

2.25. DEFINITION. For any set $\beta\subset\mathfrak{R}_n$ we define β^∇ as the set of points of the form $2x-y$, where $x\in\beta$ and $y\in\beta$.

In case β is a convex set it is easy to see that $x\in\beta^\nabla$ is equivalent to: there exists a set β' which is a translate of β, for which $\beta\cdot\beta'\neq 0$ and $x\in\beta'$.

2.26. DEFINITION. $\mathfrak{F}$ is a *hive* whenever $\mathfrak{F}$ is such a family of bounded, closed, convex subsets of $\mathfrak{R}_n$ that

(i) $\beta\in\mathfrak{F}$ implies β has interior points;

(ii) ($\beta_1\in\mathfrak{F}$ and $\beta_2\in\mathfrak{F}$) implies (β_1 either contains or is contained in some translate of β_2).

2.27. DEFINITION. F is a *convexical* blanket if and only if F is a blanket and $\mathfrak{H}$ is such a generator that if z is in the domain of F then there is a number r such that

(i) $0<r\leqq 1$;

(ii) $\alpha\in\mathfrak{H}_z$ implies $\alpha^r\subset\beta\subset\alpha$ for some $\beta\in F(z)$.

3. Some properties of convexical blankets.

For the entirety of this section we shall let F denote an arbitrary convexical blanket whose domain is A, and whose 2.27 associated generator is $\mathfrak{H}$. L and ϕ will denote, respectively, n-dimensional Lebesgue measure, and an arbitrary member of the 2.5 family $\mathfrak{B}$.

It proves convenient to introduce the function $\bar{\Phi}$, associated with ϕ, so defined that for $\beta\in\mathfrak{R}_n$, $\bar{\Phi}(\beta)$ is the inf of numbers of the form $\phi(\alpha)$, where α is such an open set that $\beta\subset\alpha$. It is easy to check[4] that $\bar{\Phi}\in\mathfrak{B}$ and

$$\text{(I)} \qquad \bar{\Phi}(\beta) = \phi(\beta)$$

whenever β is a Borel set.

[4] See A. P. Morse and J. F. Randolph, *The ϕ-rectifiable subsets of the plane*, Trans. Amer. Math. Soc. vol. 55 (1944) p. 246, Theorem 3.12.

An obvious fact which we shall have occasion to use is embodied in the following statement.

3.1. Lemma. *If β is a symmetrical convex subset of $\mathfrak{R}_n$ then $\beta^\nabla = \beta^3$.*

3.2. Lemma. *If β_0 is a symmetrical closed convex subset of $\mathfrak{R}_n$ having interior points, $1 \leqq s$, and $\mathfrak{K}_0$ is such a disjointed family of sets that $\beta \in \mathfrak{K}_0$ implies that β is a translate of β_0 for which $\beta^s \cdot \beta_0^s \neq 0$, then $\mathfrak{K}_0$ has at most $(3s)^n$ members.*

Proof. We suppose that $\mathfrak{K}_0$ has N members. We select an arbitrary set β belonging to $\mathfrak{K}_0$, recall our hypotheses, and use 2.25 and 3.1 to conclude that

$$\beta \subset \beta^{s} \subset (\beta_0^{s})^\nabla = (\beta_0^{s})^{3} = \beta_0^{3s};$$

hence from the arbitrary nature of $\beta \in \mathfrak{K}_0$ it follows that

$$\sigma\mathfrak{K}_0 \subset (\beta_0^{s})^\nabla = \beta_0^{3s};$$

$$L(\sigma\mathfrak{K}_0) \leqq L(\beta_0^{3s}) = (3s)^n L(\beta_0).$$

Thus, keeping in mind that $\mathfrak{K}_0$ is disjointed, β_0 is closed, and

$$L(\beta) = L(\beta_0)$$

whenever $\beta \in \mathfrak{K}_0$, it is evident that

$$NL(\beta_0) = \sum_{\beta \in \mathfrak{K}_0} L(\beta) = L(\sigma\mathfrak{K}_0) \leqq (3s)^n L(\beta_0).$$

Since $L(\beta_0) \neq 0$, the desired result follows at once, and the lemma is proved.

Recalling 2.22, we define for $0 < r \leqq 1$, $m > 0$, and $k > 0$,

$$E_{r,m,k} = A \cdot \underset{z}{E} \, [\phi(\alpha^9) \geqq m\phi(\alpha^r) \text{ whenever } \alpha \in \mathfrak{H}_z \text{ and } (\text{diam } \alpha) < k].$$

3.3. Theorem. *If $0 < r \leqq 1$, $m > (81/r)^n$, and $k > 0$, then $\phi(E_{r,m,k}) = 0$.*

Proof. We shall prove the theorem by contradiction; we therefore assume that $\phi(E_{r,m,k}) > 0$. We further suppose, with no loss of generality, that $E_{r,m,k}$ is bounded.

We let ϵ denote an arbitrary positive number, recall (I) above, and determine such a bounded open set D that

$$(1) \qquad E_{r,m,k} \subset D; \qquad \bar{\phi}(D) = \phi(D) \leqq \bar{\phi}(E_{r,m,k}) + \epsilon < \infty.$$

It is clear that one can find such a sequence of closed sets $C_1 \subset C_2 \subset C_3 \subset \cdots$ that $D = \sum_{i=1}^{\infty} C_i$. Since $\bar{\phi} \in \mathfrak{V}$, it follows that $\bar{\phi}(E_{r,m,k}) = \bar{\phi}(D \cdot E_{r,m,k}) = \lim_{i \to \infty} \bar{\phi}(C_i \cdot E_{r,m,k})$. Hence we may and do determine such a closed set $C \subset D$ that

$$\Phi(A_0) \geqq \Phi(E_{r,m,k}) - \epsilon, \tag{2}$$

where $A_0 = C \cdot E_{r,m,k}$.

We denote by δ the obviously positive infimum of numbers of the form $\rho(x, y)$, where $x \in C$ and $y \in (\mathfrak{R}_n - D)$, let c denote the smaller of the numbers k and δ, and select a set $\beta_0 \in \mathfrak{H}$ for which 9 (diam β_0) $< c$.

We next introduce the families $\mathfrak{F}'$, $\mathfrak{G}'$, $\mathfrak{K}'$, and $\mathfrak{J}'$ defined, respectively, by:

(i) $\alpha \in \mathfrak{F}'$ if and only if α is a set obtained by translating β_0 by the vector z, where z is some point in A_0;

(ii) $\alpha \in \mathfrak{G}'$ if and only if $\alpha = \beta^r$, where $\beta \in \mathfrak{F}'$;

(iii) $\alpha \in \mathfrak{K}'$ if and only if $\alpha = \beta^{r/3}$, where $\beta \in \mathfrak{F}'$;

(iv) $\alpha \in \mathfrak{J}'$ if and only if $\alpha = \beta^9$, where $\beta \in \mathfrak{F}'$.

There is obviously such a one-to-one correspondence between the families $\mathfrak{F}'$, $\mathfrak{G}'$, $\mathfrak{K}'$, and $\mathfrak{J}'$ that to each $\beta \in \mathfrak{F}'$ there correspond the sets β^r, $\beta^{r/3}$, and β^9 belonging, respectively, to $\mathfrak{G}'$, $\mathfrak{K}'$, and $\mathfrak{J}'$.

We easily determine by induction such a family $\mathfrak{K} \subset \mathfrak{K}'$ that

(v) $\mathfrak{K}$ is disjointed;

(vi) $\alpha \in \mathfrak{K}'$ implies that α intersects some member of $\mathfrak{K}$.

We let $\mathfrak{F}$, $\mathfrak{G}$, and $\mathfrak{J}$ denote, respectively, those subfamilies of $\mathfrak{F}'$, $\mathfrak{G}'$, and $\mathfrak{J}'$ which correspond to $\mathfrak{K}$ in the one-to-one way just mentioned.

Since $\mathfrak{K}$ is disjointed, $\sigma\mathfrak{K}$ is bounded, and

$$L(\alpha) = (r/3)^n L(\beta_0) > 0$$

for each $\alpha \in \mathfrak{K}$, it is clear that $\mathfrak{K}$, $\mathfrak{F}$, $\mathfrak{G}$, and $\mathfrak{J}$ are finite families.

The proof is completed in three steps.

Step I. $A_0 \subset \sigma\mathfrak{G} \subset \sigma\mathfrak{J} \subset D$.

Proof. That $\sigma\mathfrak{G} \subset \sigma\mathfrak{J} \subset D$ follows from the definitions (i), (ii), and (iv), together with the fact that 9 (diam β_0) $< c$. To show that $A_0 \subset \sigma\mathfrak{G}$, we assume the contrary and suppose that $z \in (A_0 - \sigma\mathfrak{G})$. In accordance with (i) and (iii) we find such a set $\beta \in \mathfrak{F}'$ that $\beta^{r/3} \in \mathfrak{K}'$ and $z \in \beta^{r/3}$. Using (vi) and 3.1 we determine such a set $\alpha \in \mathfrak{F}$ that

$$\alpha^{r/3} \in \mathfrak{K}; \qquad (\beta^{r/3})(\alpha^{r/3}) \neq 0; \qquad \beta^{r/3} \subset (\alpha^{r/3})^{\nabla} = \alpha^r.$$

Since $\alpha^r \in \mathfrak{G}$, we learn that $z \in \sigma\mathfrak{G}$, contradicting the assumption that $z \notin \sigma\mathfrak{G}$. This completes the proof of Step I.

Step II. If $0 < r \leqq 1$, $\beta \in \mathfrak{F}$, and $z \in \beta^9$, then z is a member of not more than $(81/r)^n$ members of $\mathfrak{J}$.

Proof. We define $\mathfrak{J}_0$ as that subfamily of $\mathfrak{J}$ whose members intersect β^9, and let $\mathfrak{K}_0$ be that subfamily of $\mathfrak{K}$ which corresponds to $\mathfrak{J}_0$ in the one-to-one way previously mentioned. Putting $\beta_0' = \beta^{r/3}$,

$s=27/r$, and recalling that $\mathfrak{K}_0$ is disjointed, we see that Lemma 3.2 may be applied to conclude that $\mathfrak{K}_0$ and $\mathfrak{J}_0$ each have at most $(3s)^n = (81/r)^n$ members, so that z is in at most $(81/r)^n$ members of $\mathfrak{J}$, as was to be shown.

Step III. $\phi(E_{r,m,k})=0$.

Proof. We observe with the help of Step II that

$$\sum_{\gamma\in\mathfrak{J}} K_\gamma(z) \leqq (81/r)^n$$

for each $z\in\mathfrak{R}_n$, recall that $\sigma\mathfrak{J}\subset D$, and conclude that

$$\begin{aligned}\sum_{\gamma\in\mathfrak{J}} \bar{\phi}(\gamma) &= \sum_{\gamma\in\mathfrak{J}} \int_D K_\gamma(z)d\bar{\phi}(z) \\ &= \int_D \left\{\sum_{\gamma\in\mathfrak{J}} K_\gamma(z)\right\} d\bar{\phi}(z) \leqq (81/r)^n\bar{\phi}(D).\end{aligned} \tag{3}$$

Hence, using (2), Step I, the facts that (diam β) $<k$ and $\beta\in\mathfrak{H}_z$ for some $z\in E_{r,m,k}$ whenever $\beta\in\mathfrak{F}$, the definition of $E_{r,m,k}$, (I), (3), and (1), it follows that

$$\begin{aligned} 0 &< m\phi(E_{r,m,k}) \leqq m\bar{\phi}(E_{r,m,k}) \leqq m\bar{\phi}(A_0) + m\epsilon \\ &\leqq m\bar{\phi}(\sigma\mathfrak{G}) + m\epsilon \leqq m \sum_{\beta\in\mathfrak{G}} \bar{\phi}(\beta) + m\epsilon = m\left\{\sum_{\alpha\in\mathfrak{F}} \bar{\phi}(\alpha^r)\right\} + m\epsilon \\ &\leqq \sum_{\gamma\in\mathfrak{J}} \bar{\phi}(\gamma) + m\epsilon \leqq (81/r)^n\bar{\phi}(D) + m\epsilon \\ &\leqq (81/r)^n\bar{\phi}(E_{r,m,k}) + (81/r)^n\epsilon + m\epsilon < \infty. \end{aligned} \tag{4}$$

The arbitrary nature of ϵ in (4) permits us to infer that

$$0 < m\bar{\phi}(E_{r,m,k}) \leqq (81/r)^n\bar{\phi}(E_{r,m,k}) < \infty,$$

from which, in turn, we obtain $m\leqq(81/r)^n$. This contradiction of the original hypotheses proves Step III and completes the proof of 3.3.

3.4. Theorem. *If* $0<r\leqq 1$, *then*

$$\operatorname*{limalinf}_{\mathfrak{H}_z\ni\beta\to z} \frac{\phi(\beta^9)}{\phi(\beta^r)} < \infty$$

for ϕ-almost all $z\in A$.

Proof. We let

$$A' = \mathop{E}_z\left[\operatorname*{limalinf}_{\mathfrak{H}_z\ni\beta\to z} \frac{\phi(\beta^9)}{\phi(\beta^r)} < \infty\right],$$

and select any number $m>(81/r)^n$. If $z\in(A-A')$, it is evident that either

$$\underset{\mathfrak{H}_z \ni \beta \to z}{\text{limal}} \frac{\phi(\beta^9)}{\phi(\beta^r)} = \infty$$

or else there exists such a number $k > 0$ that $\phi(\beta^r) = 0$ whenever $\beta \in \mathfrak{H}_z$ and (diam β) $< k$. It is therefore easy to see that

$$(A - A') \subset \sum_{j=1}^{\infty} E_{r,m,1/j},$$

and application of Theorem 3.3 yields $\phi(A - A') = 0$. The theorem is thus proved.

We choose a positive integer q for which

$$2 < (1 + 1/2^n)^q,$$

let $\mathfrak{A}$ be the family defined by

$$\mathfrak{A} = \sum_{z \in A} \mathfrak{H}_z,$$

and so define the function Δ with domain $\mathfrak{A}$ that

$$\Delta(\alpha) = [L(\alpha)]^q$$

for each $\alpha \in \mathfrak{A}$. Referring to Definitions 2.22, 2.23, and 2.26, we see that $\mathfrak{A}$ is a hive. Recalling 2.13 and 3.1, it follows[5] that for each $\alpha \in \mathfrak{A}$,

$$\text{(II)} \qquad \Delta : \alpha \subset \alpha^{\nabla\nabla} = \alpha^9.$$

In accordance with Definition 2.27, we may and do associate with each $z \in A$ such a number r_z, $0 < r_z \leqq 1$, that for each $\alpha \in \mathfrak{H}_z$ there exists some $\beta \in F(z)$ for which

$$\text{(III)} \qquad \alpha^{r_z} \subset \beta \subset \alpha.$$

Using Theorem 3.4 and (II), we define such a blanket H with domain A that $H(z) \subset \mathfrak{H}_z$ whenever $z \in A$ and

$$\text{(IV)} \qquad \underset{H(z) \ni \alpha \to z}{\text{limalsup}} \frac{\phi(\Delta : \alpha)}{\phi(\alpha^{r_z})} \leqq \underset{H(z) \ni \alpha \to z}{\text{limalsup}} \frac{\phi(\alpha^9)}{\phi(\alpha^{r_z})} < \infty$$

for ϕ-almost all $z \in A$.

Keeping (III) in mind, we are able to construct G, such a 2.18 sublanket of F with domain A, that to each β in the 2.10 spread of G there correspond a point z_β and a set α_β for which

$$\text{(V)} \qquad z_\beta \in A; \qquad \beta \in G(z_\beta); \qquad \alpha_\beta \in H(z_\beta); \qquad \alpha_\beta^{r_{z_\beta}} \subset \beta \subset \alpha_\beta.$$

We now so define the function Δ_0 whose domain is $\mathfrak{B}$, the spread of G, that

$$\Delta_0(\beta) = \Delta(\alpha_\beta)$$

whenever $\beta \in \mathfrak{B}$. Using (II), (V), and the obvious fact that

[5] For a proof see M, p. 215, Theorem 7.2.

$$\sum_{\beta \in \mathfrak{B}} \text{sng } \alpha_\beta \subset \mathfrak{A},$$

it is easily checked that

(VI) $$\alpha_\beta^{r_z\beta} \subset \beta \subset \Delta_0 : \beta \subset \Delta : \alpha_\beta$$

whenever $\beta \in \mathfrak{B}$. Finally, since

$$\sum_{\beta \in G(z)} \text{sng } \alpha_\beta \subset H(z)$$

whenever $z \in A$, we may use (IV) and (VI) to conclude that

$$\operatorname*{limalsup}_{G(z) \ni \beta \to z} \frac{\phi(\Delta_0 : \beta)}{\phi(\beta)} \leqq \operatorname*{limalsup}_{G(z) \ni \beta \to z} \frac{\phi(\Delta : \alpha_\beta)}{\phi(\alpha_\beta^{r_z})} \leqq \operatorname*{limalsup}_{H(z) \ni \alpha \to z} \frac{\phi(\Delta : \alpha)}{\phi(\alpha^{r_z})} < \infty$$

for ϕ-almost all $z \in A$, which 2.14 leads to the following result.

3.5. THEOREM. *There is a ϕ-regular subblanket of F with domain A.*

A redefinition of convexical blankets might be effected by deleting the word "symmetrical" in (i) of 2.23, as it applies to the definition of the generator $\mathfrak{H}$, and in turn F itself. However, it happens that such a change would bestow no additional generality upon F or the results obtained in Theorem 3.5; for if F were defined in terms of such a nonsymmetrical generator $\mathfrak{H}$, then in accordance with well known properties[6] of convex sets, a suitable symmetrical generator $\mathfrak{H}'$ could be defined in terms of $\mathfrak{H}$ which would still satisfy the requirements of Definition 2.27 and which could therefore serve as generator for F in place of $\mathfrak{H}$.

Although Theorem 3.5 guarantees the existence of a ϕ-regular subblanket of F with domain A, nevertheless, in order that the subblanket possess suitable properties for purposes of differentiation, it is necessary that F satisfy an additional requirement not mentioned in Definition 2.27. Specifically, we need to require that F be a 2.12 Borelian or close blanket. If F is Borelian, then our 3.5 subblanket is suitable for differentiation by virtue of M, Theorems 11.2, 11.3, and 8.12. If F is a close blanket, we can say even more, since from 2.20 we can then conclude that our 3.5 subblanket is 2.19 ϕ-strong, and hence M, Theorem 8.10 holds for the subblanket.

4. **A counterexample.** If $\phi \in \mathfrak{B}$ and $\mathfrak{H}$ is a generator, then in view of Theorem 3.4 and (IV) above, one might ask: For each r, $0 < r \leqq 1$, must there exist a generator $\mathfrak{H}' \subset \mathfrak{H}$ for which

[6] Use T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*, Ergebnisse der Mathematik, vol. 3, 1934, p. 53, formula (1).

$$\operatorname*{limalsup}_{\mathfrak{H}'_z \ni \beta \to z} \frac{\phi(\beta^q)}{\phi(\beta^r)} < \infty$$

for ϕ-almost all $z \in \mathfrak{R}_n$? It turns out that the answer to this question is negative. We shall construct a certain example in the straight line space $\mathfrak{R}_1$, and outline a proof of this last statement. Henceforth, ρ will be the usual metric for $\mathfrak{R}_1$, and such terms as length, closed, open, and so on, will have reference to ρ.

We suppose that I is a closed interval in $\mathfrak{R}_1$ of length $l > 0$, and let n be a positive integer. We so choose $2n-1$ equally spaced points $P_1, P_2, \cdots, P_{2n-1}$ in the interior of I that they divide I into $2n$ equal parts. Each odd numbered point $P_1, P_3, \cdots, P_{2n-1}$ is made the center of a closed interval of length l/n^2. The n reduced intervals thus constructed are evidently disjointed and all are contained in I itself. We let 0 be that function for which $0(n, m, I)$ is the mth such interval counting from the right, whenever m is such a positive integer that $1 \leqq m \leqq n$.

We begin our construction by selecting a closed interval I_0 in $\mathfrak{R}_1$, of unit length. Recalling 2.7, we define by induction the families

$$\mathfrak{F}_0 = \operatorname{sng} I_0; \qquad \mathfrak{F}_n = \sum_{I \in \mathfrak{F}_{n-1}} \left\{ \sum_{m=1}^{n} \operatorname{sng} 0(n, m, I) \right\},$$

for $n = 1, 2, 3, \cdots$; we define $\mathfrak{F} = \sum_{n=0}^{\infty} \mathfrak{F}_n$. It is clear that $\sigma\mathfrak{F}_1 \supset \sigma\mathfrak{F}_2 \supset \sigma\mathfrak{F}_3 \supset \cdots$. We define the closed set $A_0 = \prod_{n=0}^{\infty} (\sigma\mathfrak{F}_n)$. It is to be noted that if n is a positive integer and $I \in \mathfrak{F}_n$, then there exists precisely one integer m and one interval $I' \in \mathfrak{F}_{n-1}$, such that $1 \leqq m \leqq n$ and $I = 0(n, m, I')$. We define inductively the function χ with domain $\mathfrak{F}$ for which

$$\chi(I_0) = 1; \qquad \chi\{0(n, m, I)\} = \frac{\lambda_n}{m^m} \chi(I)$$

whenever $I \in \mathfrak{F}_{n-1}$, m is such a positive integer that $1 \leqq m \leqq n$, and

$$\lambda_n = 1 \Big/ \sum_{j=1}^{n} \frac{1}{j^j} \cdot$$

From this definition we see that

$$\sum_{m=1}^{n+1} \chi\{0(n+1, m, I)\} = \chi(I)$$

whenever $I \in \mathfrak{F}_n$. Applying induction to this last equation, it is easy to see that if $\mathfrak{G}_n \subset \mathfrak{F}_n$, n' is any integer exceeding n, and

$$\mathfrak{G}_{n'} = \mathfrak{F}_{n'} \cdot E_I [I \subset \sigma\mathfrak{G}_n],$$

then

(i)
$$\sum_{t\in\mathfrak{G}_n}\chi(t) = \sum_{t'\in\mathfrak{G}_n'}\chi(t').$$

We define $\mathfrak{O}$ as the family of open subsets of $\mathfrak{R}_1$. For each positive integer n, we define ω_n as the function on $\mathfrak{O}$ for which

$$\omega_n(T) = \sum_{I\in\mathfrak{H}_n}\chi(I),$$

whenever $T\in\mathfrak{O}$ and

$$\mathfrak{H}_n = \mathfrak{F}_n\cdot \underset{I}{E}\,[I\subset T].$$

It is clear from (i) that $\omega_{n+1}(T)\geqq\omega_n(T)$ for each such n and T. We are therefore able to define that function Ω with domain $\mathfrak{O}$ for which

$$\Omega(T) = \lim_{n\to\infty}\omega_n(T)$$

whenever $T\in\mathfrak{O}$. Finally, we define Ω_0 as that function whose domain is the family of subsets of $\mathfrak{R}_1$, such that for $\beta\subset\mathfrak{R}_1$,

$$\Omega_0(\beta) = \inf_{T\in\mathfrak{G}}\Omega(T),$$

where

$$\mathfrak{G} = \mathfrak{O}\cdot\underset{T}{E}\,[\beta\subset T].$$

The following unproved statements are consequences of the foregoing definitions and observations.

(ii) If $I\in\mathfrak{F}$, then $\Omega_0(I)=\chi(I)$.

(iii) Ω_0 is a member of the 2.5 family $\mathfrak{V}$.

(iv) $\Omega_0(A_0)=1$.

For $n=0, 1, 2, 3, \cdots$ we let l_n denote the length of an interval belonging to $\mathfrak{F}_n$. We also define β_n as that interval with center at the origin of length

$$l_n\left(\frac{1}{n+1}+\frac{2}{(n+1)^2}\right).$$

We let $\mathfrak{H}$ be the 2.23 generator whose members are $\beta_0, \beta_1, \beta_2, \cdots$. It can be shown that if $0<r<1$ and $\mathfrak{H}'$ is any generator contained in $\mathfrak{H}$, then

$$\operatorname*{limalsup}_{\mathfrak{H}_z'\ni\beta\to z}\frac{\Omega_0(\beta)}{\Omega_0(\beta^r)} = \infty$$

for Ω_0-almost all $z\in A_0$. This result is more than sufficient to justify the assertion made at the beginning of this section.

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