THE CLASSICAL EXISTENCE THEOREM OF NONLINEAR ANALYTIC DIFFERENTIAL EQUATIONS

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Let $f_1, \cdots, f_n$, where $f_j = f_j(s, w_1, \cdots, w_n)$, be n power series in $w_1, \cdots, w_n$,

$$(1) \qquad f_j = \sum_{k=0}^{\infty} \cdots \sum_{m=0}^{\infty} a_j^{k\cdots m}(s) w_1^k \cdots w_n^m,$$

where the coefficients $a(s)$ are uniformly almost periodic functions on some half-plane, say,

$$(2) \qquad \sigma > 0 \qquad (s = \sigma + it),$$

which possess Fourier expansions

$$(3) \qquad a(s) \sim \sum_p \alpha_p e^{-\lambda_p s},$$

where the exponents λ_p are all positive and bounded away from zero,

$$(4) \qquad \lambda_p \geqq \lambda_0 > 0.$$

Further, let $A_j^{k\cdots m} < \infty$, where

$$(5) \qquad A_j^{k\cdots m} = \operatorname*{l.u.b.}_{\sigma>0} \left| a_j^{k\cdots m}(s) \right|,$$

and assume that for some, sufficiently small, positive r,

$$(6) \qquad \sum_{k=0}^{\infty} \cdots \sum_{m=0}^{\infty} A_j^{k\cdots m} r^{k+\cdots+m} < \infty .$$

In particular, (6) implies that the functions $f_1, \cdots, f_n$ are all regular in the domain

$$(7) \qquad \sigma > 0, \ |w_1| < r, \cdots, |w_n| < r.$$

In these notations, the following theorem will be proved:

THEOREM. *In the domain* (7), *let* $f_1, \cdots, f_n$ *be regular functions of the form* (1) *with uniformly almost periodic coefficient functions defined on* (2) *having Fourier expansions of the type* (3), *satisfying* (4). *Then, if* (6) *holds, there exists a half-plane* $\sigma > l \geqq 0$ *on which the system of differential equations*

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$$dw_j/ds = f_j(s, w_1, \cdots, w_n) \qquad (j = 1, \cdots, n) \tag{8}$$

possesses a uniformly almost periodic solution vector $w_1(s), \cdots, w_n(s)$ *satisfying the "initial conditions"* $w_j(s) \to 0$ *as* $\sigma \to \infty$, $j = 1, \cdots, n$. *In addition, this uniformly almost periodic solution is the only solution satisfying the latter "initial conditions."*

The latter initial conditions will be proved by showing that a uniformly almost periodic solution has exponents which are minorized by the positive constant λ_0 occurring in (4). It will also be clear from the proof that the exponents occurring in the solution are linear combinations, with non-negative integral coefficients, of the exponents appearing in the expansions (3) of the coefficients of (1).

This theorem is an extension of the classical existence theorem for analytic systems of differential equations,

$$dw_j/dz = F_j(z, w_1, \cdots, w_n) \qquad (j = 1, \cdots, n)$$

with the initial condition $w_1(0) = 0, \cdots, w_n(0) = 0$ (z corresponds to e^{-s}). Another extension of the classical existence theorem was considered in [2],[1] for the case where the coefficient functions $a(s)$ in (1) are all absolutely convergent Laplace-Stieltjes transforms with a positive lower limit of integration,

$$a(s) = \int_{\lambda_0}^{\infty} e^{-s\lambda} d\alpha(\lambda), \qquad \lambda_0 > 0,$$

where $\sigma > 0$. Neither of these extensions of the classical theorem contains the other. The latter extension had to be proved by using the integral analogue of the method of comparison of coefficients and convolution theory, whereas the present proof will depend on an application of the method of successive approximations. The latter method was used in [3] for that particular case of the theorem where the system (8) becomes linear, a case in which, of course, more information was obtained concerning the existence region of the solution than in the present nonlinear case.

For simplicity the theorem will be proved for the scalar case, $n = 1$. Since the proof will depend upon majorants in the method of successive approximations, it will be clear that this is no loss of generality.

For $n = 1$, (8) reduces to

$$w' = f(s, w) \qquad (' = d/ds), \tag{9}$$

where

[1] Numbers in brackets refer to the references at the end of the paper.

$$f(s, w) = \sum_{m=0}^{\infty} a_m(s) w^m, \tag{10}$$

and, by virtue of (3) and (4), the $a_m(s)$ are uniformly almost periodic functions on (2) with Fourier expansions

$$a_m(s) \sim \sum_{p}^{m} \alpha_p e^{-\lambda_p s}, \qquad \lambda_p \geqq \lambda_0 > 0. \tag{11}$$

The assumption (6) is that for some, sufficiently small, $r>0$,

$$\sum_{m=0}^{\infty} A_m r^m < \infty , \tag{12}$$

where

$$A_m = \operatorname*{l.u.b.}_{\sigma>0} | a_m(s) | . \tag{13}$$

The assertion is that (12) implies that there exists a half-plane $\sigma > l \geqq 0$ on which (9) has a unique uniformly almost periodic solution $w(s)$ for which $w(s) \to 0$, $\sigma \to \infty$.

As to the value of l, the following information will be obtained: *If the lower bound* λ_0 *occurring in* (11) *is assumed to be* 1, *and the sum of the series* (12) *for* $r<1$ *to be less than* 1, *then* l *can be chosen to be* 0. The proof of this will depend upon the use of Bohr's compactness theorem of normal families of regular, uniformly almost periodic functions. In fact, the argument is the same as that used in [**5**] to obtain the best value of the radius of regularity of the solution of an implicit analytic system. It follows from examples, even in the classical case of ordinary power series (cf. [**4**]), that these values of the constants can not be improved.

Proof of the theorem. The exponents λ_p can be assumed to satisfy

$$\lambda_p \geqq 1 \tag{14}$$

in (11), for it is assumed that $\lambda_p \geqq \lambda_0$ holds for all p, and the assumptions and assertions of the theorem remain unchanged if s is replaced everywhere by s/λ_0. Also it is no essential restriction on (12) to assume that $r<1$.

The assumption (14) implies that the functions $a_m(s)e^s$ are uniformly almost periodic on $\sigma>0$ with non-negative exponents, and consequently tend to finite limits as $\sigma \to \infty$ (cf. [**1**, p. 152]). This implies, by (13), that $|a_m(s)| \leqq A_m e^{-\sigma}$, $\sigma > 0$. In particular, if $A_m(\sigma_0)$ denotes the least upper bound of $|a_m(s)|$ on the half-plane

$\sigma > \sigma_0 (A_m(0) = A_m)$, it follows that $A_m(\sigma_0) \leqq A_m e^{-\sigma_0}$. Therefore the series

$$\sum_{m=0}^{\infty} A_m(\sigma_0) r^m$$

can be made to have a sum less than 1 if σ_0 is chosen sufficiently large, and $r < 1$. Since the theorem asserts the existence of a solution only on *some* half-plane $\sigma > l \geqq 0$, it is clear, in view of the foregoing remarks, that it can be assumed at the outset that

$$\sum_{m=0}^{\infty} A_m r^m < 1, \quad r < 1. \tag{15}$$

Define the successive approximations $w_0(s)$, $w_1(s)$, $\cdots$ as follows:

$$w_0(s) \equiv 0, \quad w'_{k+1}(s) = f(s, w_k(s)), \qquad k = 0, 1, 2, \cdots, \tag{16}$$

and

$$\lim_{\sigma \to \infty} w_{k+1}(s) = 0. \tag{17}$$

First, it should be noted that if $w_k(s)$ has been defined in the half-plane $\sigma > 0$ in such a way that it is uniformly almost periodic and satisfies

$$| w_k(s) | < 1, \quad \sigma > 0, \tag{18}$$

and is such that all of its exponents satisfy (14), then (16) and (17) determine in the half-plane $\sigma > 0$ a uniformly almost periodic function $w_{k+1}(s)$ for which (18) is true when k is replaced by $k+1$, and for which the exponents satisfy (14). The truth of this statement follows from a lemma used in [3, p. 861] concerning the indefinite integral of a bounded uniformly almost periodic function on $\sigma > 0$. In fact if (18) is satisfied, then by (15), $|f(s, w_k)| < 1$, and that lemma asserts that $w_{k+1}(s)$ is uniformly almost periodic on $\sigma > 0$, $|w_{k+1}(s)| < e^{-\sigma} < 1$, $\sigma > 0$, by virtue of (14), and the Fourier exponents of $w_{k+1}(s)$ are those of $f(s, w_k)$.

It will now be shown that the series

$$\sum_{k=0}^{\infty} \{ w_{k+1}(s) - w_k(s) \} \tag{19}$$

is uniformly convergent on the half-plane $\sigma > 0$. In order to show this, note that (15) implies that the partial derivative $f_w(s, w)$ is bounded on every half-plane $\sigma \geqq \delta > 0$ and, from (10) and (11), $f_w(s, w) \to 0$ as $\sigma \to \infty$, uniformly in t, for all $|w| < 1$ (cf. [1, p. 151]). Therefore, for every constant $c<1$, there exists an $l\geqq 0$ such that

$$|f_w(s, w)| \leqq c < 1, \qquad \sigma > l, \ |w| < 1. \tag{20}$$

From this, and since $f(s, w)$ is a power series in w for which (15) is true, it follows that $f(s, w)$ satisfies a Lipschitz condition, that is, the inequality

$$|f(s, w^*) - f(s, w^{**})| \leqq c|w^* - w^{**}| \qquad (c < 1) \tag{21}$$

holds whenever (s, w^*), (s, w^{**}) are a pair of points in the domain

$$\sigma > l, \quad |w| < 1. \tag{22}$$

By virtue of (16) and (18) this implies the inequality

$$|w'_{k+1}(s) - w'_k(s)| \leqq c|w_k(s) - w_{k-1}(s)|, \qquad \sigma > l. \tag{23}$$

If

$$B_k = \underset{\sigma>l}{\text{l.u.b.}} |w_k(s) - w_{k-1}(s)|, \tag{24}$$

then from (23), (14), and the lemma referred to after (18), it follows that

$$|w_{k+1}(s) - w_k(s)| \leqq cB_k, \qquad \sigma > l. \tag{25}$$

This implies that

$$B_{k+1} \leqq cB_k. \tag{26}$$

Since $|w_1(s)-w_0(s)| \leqq A_0e^{-\sigma}$, $\sigma>0$, $B_1 \leqq A_0e^{-l} \leqq A_0$, and therefore an induction on (26) shows that

$$B_{k+1} \leqq A_0c^k. \tag{27}$$

But $c<1$, $\sigma>l$, and therefore (27), by virtue of (24), proves that the series (19) is absolutely-uniformly convergent for $\sigma>l$. Its sum, $w(s)$, is uniformly almost periodic for $\sigma>l$, since every term of (19) is.

To say that the series (19) is uniformly convergent is to say that $w_k(s)\rightarrow w(s)$, $k\rightarrow\infty$, uniformly for $\sigma>l$. But the sequence of regular functions $w_1(s)$, $w_2(s)$, $\cdots$ is uniformly bounded for $\sigma>0$ (cf. (18)). Therefore, the $w_k(s)$ tend uniformly to the regular function $w(s)$ on every compact subset of $\sigma>0$. Furthermore, from (18), $|w(s)|\leqq 1$, $\sigma>0$. By a standard theorem in almost periodic functions (cf. [1, pp. 142, 143]) it follows, since $w(s)$ is uniformly almost periodic on $\sigma>l\geqq 0$, regular and bounded on $\sigma>0$, that $w(s)$ is uniformly almost periodic on the half-plane $\sigma>0$.

Furthermore, (25), (23), (16), (17) show that the derived series

$$\sum_{k=0}^{\infty} \left\{ w'_{k+1}(s) - w'_k(s) \right\}$$

is uniformly convergent for $\sigma > 0$. Therefore $w(s)$ is differentiable on $\sigma > 0$ and, by virtue of (16), satisfies (9).

From (10) and (11) it follows that if a solution $w = w(s)$ of (9) tends to 0, as $\sigma \to \infty$, then $f(s, w(s)) = O(e^{-\sigma})$, $\sigma \to \infty$ (cf. (14)). This implies, by virtue of (9), that $w' = O(e^{-\sigma})$, $\sigma \to \infty$, and consequently $w = O(e^{-\sigma})$, $\sigma \to \infty$. Therefore, the last assertion of the theorem, concerning the uniqueness of the solution, follows from standard arguments used for differential equations satisfying a Lipschitz condition.

References

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