ON THE DEFINING FIELD OF A DIVISOR IN AN ALGEBRAIC VARIETY[1]

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In the system of algebraic geometry as developed by A. Weil in his recent book, *Foundations of algebraic geometry*, a variety U in the n-space is defined as the set of all equivalent couples (k, P), each consisting of a field k and a point P in the n-space such that the field $k(P)$ is a regular extension of k. Two such couples (k', P') and (k'', P'') are called equivalent if every finite specialization of P' over k' is also one of P'' over k'' and conversely. Any field k which enters into such a couple is called a field of definition of the variety U. It has been shown by Weil in his book that among all the fields of definition of a variety U there is a smallest one which is contained in all of them, which we shall call the defining field of the variety U. A d-cycle G in a variety U of dimension r is a finite set of simple subvarieties of dimension d in U, to each of which is assigned an integer called its multiplicity; a cycle is called positive if the multiplicity of each of its component varieties is positive. Let K be a field of definition of U. Then the G is said to be rational over K if it satisfies the following conditions: (1) each component variety of G is algebraic over K; (2) if a variety is a component of G, then all the conjugate varieties over K are also components of G with the same multiplicity; (3) the multiplicity of each component of G is a multiple of its order of inseparability. The question arises whether there is a smallest one among all the fields over which the cycle G is rational. If such a smallest field exists, we shall call it the defining field of the cycle G. One observes that since by definition every field over which the cycle G is rational must be a field of definition of the variety U, it follows that the defining field of G, if it exists, must contain the defining field of U.

The following simple example shows that in general a cycle does not have a defining field. Consider the variety U_0 consisting of the one point $(x^{1/p}, y^{1/p})$ in the 2-space S_2, where x and y are independent variables over a field k of characteristic p; this variety U_0 is a simple subvariety of S_2. The 0-cycle pU_0 is rational over both fields $k(x, y^{1/p})$ and $k(x^{1/p}, y)$; but it is not rational over the field $k(x, y^{1/p})\cap k(x^{1/p}, y) = k(x, y)$.

In a recent discussion, Weil has communicated to me the conjecture

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[1] This note is essentially an extract from a letter of the author to Professor André Weil, dated February 3, 1949.

that *in case of a divisor in a variety there always exists a defining field.* I shall give here a proof of this conjecture, based on the method of the associated forms.[2] Let U be a variety of dimension r in the n-space and let k be its defining field. Let $F(Z, Z^{(1)}, \cdots, Z^{(r)})$ be the associated form of U, where the $(Z), (Z^{(1)}), \cdots, (Z^{(r)})$ are $r+1$ sets of $n+1$ indeterminates each. It can be easily shown that this form $F(Z, Z^{(1)}, \cdots, Z^{(r)})$ is rational over k and absolutely irreducible. We can restrict ourselves to positive cycles in U, since every cycle rational over a field K is the difference of two positive cycles which are rational over K. To each positive d-cycle G in U (since it is also a positive d-cycle in the n-space) there corresponds an associated form $f(Z, Z^{(1)}, \cdots, Z^{(d)})$; let (f) be the set of coefficients of this form. Then the field $k((f))$ is contained in every field over which the cycle G is rational.[3] It is clear that if the cycle G is rational over $k((f))$, then this field is its defining field. Hence, to prove our conjecture, it is necessary only to show that in case $d=r-1$ the divisor G is rational over the field $k((f))=K$.

Without loss of generality we can assume that the form $f(Z, Z^{(1)}, \cdots, Z^{(r-1)})$ is irreducible over K; for otherwise we need only to repeat the same argument for each irreducible factor of this form over K. Over the algebraic closure of K the form $f(Z, Z^{(1)}, \cdots, Z^{(r-1)})$ will then be the p^eth power of a product of distinct irreducible forms, all conjugate with respect to each other over K; this p^e is by definition the multiplicity of each component of the divisor G. Let (x) be a generic point of one of the components of G over K (it does not matter which, as they are all conjugates over K). We have to prove that the field $K(x)$ has the order of inseparability $\leqq p^e$ over K.

Let $z_j^{(i)}$, $i=1, \cdots, r$, $j=1, \cdots, n$, be nr independent variables over $K(x)$, and set $z_0^{(i)} = -\sum_{j=1}^{n} x_j z_j^{(i)}$. Since the point (x) has the dimension $r-1$, the $r(n+1)-1$ elements $z_j^{(i)}$, for all i, j except $i=r$, $j=0$, are independent variables over K. Hence, the form $f(Z, z^{(1)} \cdots, z^{(r-1)})$ is exactly the p^eth power of a product of distinct linear forms. Let $f(Z_0)$ be the polynomial obtained from $f(Z, z^{(1)}, \cdots, z^{(r-1)})$ by

[2] See Chow and van der Waerden, Math. Ann. vol. 113 (1937) pp. 692–704. Strictly speaking, the concept of the associated form is defined only for varieties and positive cycles in a projective space. However, since the affine n-space can be extended to a projective space and every variety in the former can be extended to a variety in the latter in a unique way, it is easily seen that our application of this concept to varieties and positive cycles in the affine n-space is justified.

[3] If $(f)=(f_0, f_1, \cdots, f_t)$ is a set of quantities, not all zero, then the field $K((f))$ is the extension of K obtained by the adjunction of all the ratios of any two non-vanishing elements of the set (f).

setting $Z_j = z_j^{(r)}$, $j = 1, \cdots, n$; then $f(Z_0)$ is the p^eth power of a separable polynomial. Since $z_0^{(r)}$ is a root of this polynomial $f(Z_0)$, it follows that $z_0^{(r)}$ has the order of inseparability p^e over the field $K(z^{(1)}, \cdots, z^{(r-1)}; z_1^{(r)}, \cdots, z_n^{(r)})$. Denote by K^* the field obtained from K by the adjunction of the nr element $z_j^{(i)}$, $i = 1, \cdots, r$, $j = 1, \cdots, n$. Then our result shows that $K(z^{(1)}, \cdots, z^{(r)})$ has the order of inseparability not greater than p^e over K^*. Consider now the form $F(Z, z^{(1)}, \cdots, z^{(r)})$; it dissolves over the algebraic closure of $K(z^{(1)}, \cdots, z^{(r)})$ into a product of distinct linear forms each of which corresponds to an intersection of U with the linear variety defined by the equations $\sum_{j=0}^{n} z_j^{(i)} X_j = 0$, $i = 1, \cdots, r$. For, the point (x) is a simple point of U (by the definition of a divisor) and the linear variety is a generic one containing the point (x). Since the point (x) is one of these intersections, therefore the linear form $\sum_{j=0}^{n} x_j Z_j$ is a simple factor of the form $F(Z, z^{(1)}, \cdots, z^{(r)})$; it is, in fact, the only factor independent of the variables $z_j^{(i)}$, $i = 1, \cdots, r$; $j = 1, \cdots, n$. Therefore, the point (x) is rational over the field $K(z^{(1)}, \cdots, z^{(r)})$, and hence $K^*(x) = K(z^{(1)}, \cdots, z^{(r)})$. This shows that $K^*(x)$ has the order of inseparability not greater than p^e over K^*; and since $K(x)$ and K^* are linearly disjoint over K, it follows that $K(x)$ has also the order of inseparability not greater than p^e over K.

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