the equation $\sum_{m=0}^{s} A_m \cdot \times K_m X^m = 0$, if the matrices involved are square.

Results similar to those in this paper may be obtained, in some cases more readily, in the consideration of $\sum_{m=0}^{s} (X^m K_m) \cdot \times A_m = 0$, $\sum_{m=0}^{s} A_m \times \cdot (K_m X^m) = 0$, and $\sum_{m=0}^{s}(X^m K_m) \times \cdot A_m = 0$, and so on, where $A \times \cdot B = (a_{ij}B)$.

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Michigan State College

AVERAGES OF CHARACTER SUMS

P. T. BATEMAN AND S. CHOWLA

Suppose that χ is a primitive residue character[1] modulo k, $k>1$, and that for y non-negative, $S(y) = \sum_{0 \leq l \leq y} \chi(l)$. It is important [see, for example, **11**] in the analytic theory of numbers to have as much information as possible about the sums $S(y)$, in particular about their maximum order of magnitude; it is known (cf. [**13**; **14**; **8**]), for example, that $S(y) < k^{1/2} \log k$, but unknown whether or not $M(\chi) = o(k^{1/2} \log k)$ as k tends to infinity, where $M(\chi)$ is the maximum of $|S(1)|, \cdots, |S(k-1)|$. Hua [**4**; **5**; **6**] has shown that it is often helpful to consider the averages $n^{-1}\sum_{m=0}^{n} S(m)$. In this paper we consider some further developments of this idea.

1. **Preliminaries.** We recall [**7**, pp. 483–486, 492–494] that if χ is a primitive residue character mod k and if $\tau(\chi) = \sum_{n=1}^{k} \chi(n) e^{2\pi i n/k}$, then $|\tau(\chi)| = k^{1/2}$ and

$$\sum_{n=1}^{k} \chi(n) e^{2\pi i m n/k} = \bar{\chi}(m)\tau(\chi) \tag{1}$$

for any integer m, $\bar{\chi}$ being the complex conjugate of χ.

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[1] For the basic facts about residue characters see [**7**, pp. 401–414, 478–494]. Numbers in brackets refer to the bibliography.

The function $S(kx)$ is of bounded variation and period unity, and as such has an everywhere convergent Fourier series. It may be deduced from (1) (cf. [13, pp. 23–24] and [8, pp. 81–82]) that this Fourier series is

$$A(\chi) - \frac{\tau(\chi)}{2\pi i} \sum_{m \neq 0} \frac{\bar{\chi}(m)}{m} e^{-2\pi i m x}, \tag{2}$$

where m runs over the positive and negative integers and

$$A(\chi) = \int_0^1 S(kx)dx = \frac{1}{k} \sum_{m=0}^{k-1} S(m). \tag{3}$$

Thus if we define a function $S^*(y)$ as $S(y)$ if y is not an integer and as $S(y) - \chi(y)/2$ if y is an integer, we have

$$S^*(kx) = \begin{cases} A(\chi) + \dfrac{\tau(\chi)}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \sin 2\pi n x & \text{if } \chi(-1) = 1 \\ A(\chi) - \dfrac{\tau(\chi)}{\pi i} \displaystyle\sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \cos 2\pi n x & \text{if } \chi(-1) = -1 \end{cases} \tag{4}$$

for all non-negative x.

2. **Another proof of Paley's Ω-result for $S(n)$.** If we put $x=0$ in (4) we get a formula for the arithmetic mean $A(\chi)$ of $S(0), \cdots, S(k-1)$, namely

$$A(\chi) = \begin{cases} 0 & \text{if } \chi(-1) = 1 \\ \dfrac{\tau(\chi)}{\pi i} L(1, \bar{\chi}) & \text{if } \chi(-1) = -1, \end{cases} \tag{5}$$

where $L(s, \bar{\chi} = \sum_{n=1}^{\infty} \bar{\chi}(n) n^{-s}$ for $R(s) > 0$. The second half of formula (5) can of course be proved by the method used in [9] to prove the second part of Satz 217 (of which the second half of (5) is a generalization), but some use of Fourier series is essential.[2]

The second part of (5) enables us, when $\chi(-1) = -1$, to use information about the order of magnitude of $L(1, \bar{\chi})$ to get information about the order of magnitude of $A(\chi)$. For example, a slight change in the argument of [2] shows that for real primitive χ with $\chi(-1) = -1$ we have

$$\limsup_{k \to \infty} \frac{L(1, \chi)}{\log \log k} \geqq e^{\gamma}, \tag{6}$$

[2] Naturally (5) can be proved in a roundabout way by showing that both sides are equal to $L(0, \chi)$. The argument indicated here essentially stems from Minkowski (cf. [13, p. 26]).

γ being Euler's constant. (The lim sup is unambiguous, since for each k there is at most one real primitive χ with $\chi(-1)=-1$.) Now $\tau(\chi)=ik^{1/2}$ for real primitive χ with $\chi(-1)=-1$ [9, Satz 215] and thus (6) gives the following result.

THEOREM 1. *If χ runs through all real primitive characters with $\chi(-1)=-1$ in order of increasing size of the modulus k of χ and if $A(\chi)$ is as defined in* (3), *then*

$$\limsup_{k\to\infty}\frac{A(\chi)}{k^{1/2}\log\log k}\geqq\frac{e^{\gamma}}{\pi}.$$

This theorem is a slightly stronger form of Paley's result (see [12; 10; 1]) that if χ is a primitive character modulo k, then $M(\chi)=\Omega(k^{1/2}\log\log k)$, where $M(\chi)$ is the maximum of $|S(0)|,\cdots,|S(k-1)|$; for our theorem shows that this Ω-result is true even for the arithmetic mean of $S(0),\cdots,S(k-1)$.

We remark in passing that Parseval's formula applied to (2) or (4) gives

$$\begin{aligned}\frac{1}{k}\sum_{m=0}^{k-1}|S(m)-A(\chi)|^2&=\int_0^1|S(kx)-A(\chi)|^2dx\\&=\frac{k}{2\pi^2}\sum_{n=1}^{\infty}{}'\frac{1}{n^2}=\frac{k}{12}\prod_{p|k}\left(1-\frac{1}{p^2}\right),\end{aligned}$$

where the $'$ indicates summation over the positive integers relatively prime to k. Alternatively we have, in view of (5),

$$\frac{1}{k}\sum_{m=0}^{k-1}|S(m)|^2=\begin{cases}\dfrac{k}{2\pi^2}\displaystyle\sum_{n=1}^{\infty}{}'\frac{1}{n^2} & \text{if}\quad\chi(-1)=1\\[2ex]\dfrac{k}{\pi^2}\left(\dfrac{1}{2}\displaystyle\sum_{n=1}^{\infty}{}'\frac{1}{n^2}+|L(1,\chi)|^2\right) & \text{if}\quad\chi(-1)=-1.\end{cases}$$

3. **Estimation of certain partial averages of $S(n)$.** We saw above that if χ is a primitive character with $\chi(-1)=1$, then the arithmetic mean of the numbers $S(0),\cdots,S(k-1)$ is zero. Hence in this case we should expect a fairly good estimate of the arithmetic mean of $S(0),\cdots,S(n-1)$ for $n<k$. And in fact Hua [4; 5; 6] has proved that we have

$$\left|\frac{1}{n}\sum_{m=0}^{n-1}S(m)\right|\leqq\frac{1}{2}\left(k^{1/2}-\frac{n}{k^{1/2}}\right)\quad\text{for}\quad 0\leqq n\leqq k,\ \chi(-1)=1.\tag{7}$$

If χ is a primitive character with $\chi(-1)=-1$, such a neat result cannot be expected, since the arithmetic mean of $S(0), \cdots, S(k-1)$ is not zero. However we prove the following result, which is valuable for n not too small relative to k, especially if $|L(1, \chi)|$ is large.

THEOREM 2. *If χ is a primitive character modulo k with $\chi(-1)=-1$ and if $\alpha k \leqq n \leqq k$, where α is a number between* 0 *and* 1, *then*

$$\left| \frac{1}{n} \sum_{m=0}^{n-1} S(m) - \frac{\tau(\chi)}{\pi i} L(1, \bar{\chi}) \right| < \left(2.1 + \frac{\pi}{4} \log \frac{1}{\alpha} \right) k^{1/2}.$$

PROOF. Put $q=[k/2]$, $\rho = e^{2\pi i/k}$. Then by (1) we have

$$\begin{aligned}
\tau(\bar{\chi})S(m) &= \sum_{l=0}^{m} \chi(l)\tau(\bar{\chi}) = \sum_{l=0}^{m} \sum_{h=-q}^{q} \bar{\chi}(h)\rho^{hl} \\
&= \sum_{h=1}^{q} \bar{\chi}(h) \sum_{l=0}^{m} (\rho^{hl} - \rho^{-hl}) \\
&= \sum_{h=1}^{q} \bar{\chi}(h) \frac{\rho^{h(m+1)} - 1 - \rho^{h} + \rho^{-hm}}{\rho^{h} - 1} \\
&= \sum_{h=1}^{q} \bar{\chi}(h) \frac{\rho^{h(m+1)} + \rho^{-hm}}{\rho^{h} - 1} \\
&\quad + i \sum_{h=1}^{q} \bar{\chi}(h) \cot \frac{\pi h}{k} \cdot
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\tau(\bar{\chi})}{n} \sum_{m=0}^{n-1} S(m) &= \frac{1}{n} \sum_{h=1}^{q} \bar{\chi}(h) \frac{\rho^{h(n+1)} - \rho^{-h(n-1)}}{(\rho^{h} - 1)^2} \\
&\quad + i \sum_{h=1}^{q} \bar{\chi}(h) \cot \frac{\pi h}{k} \\
&= \frac{1}{2in} \sum_{h=1}^{q} \bar{\chi}(h) \frac{\sin (2\pi hn/k)}{\sin^2 (\pi h/k)} \\
&\quad + i \sum_{h=1}^{q} \bar{\chi}(h) \cot \frac{\pi h}{k} \cdot
\end{aligned}$$

Now $\tau(\chi)\tau(\bar{\chi}) = -\tau(\chi)\overline{\tau(\chi)} = -k$ for $\chi(-1)=-1$ and thus

$$\begin{aligned}
&\tau(\bar{\chi}) \left\{ \frac{1}{n} \sum_{m=0}^{n-1} S(m) - \frac{\tau(\chi)}{\pi i} L(1, \bar{\chi}) \right\} \\
&\qquad = \frac{\tau(\bar{\chi})}{n} \sum_{m=0}^{n-1} S(m) - \frac{ik}{\pi} L(1, \bar{\chi})
\end{aligned}$$

$$
\begin{aligned}
(8)\qquad & = \frac{1}{2in} \sum_{h=1}^{q} \bar{\chi}(h) \frac{\sin (2\pi hn/k)}{\sin^2 (\pi h/k)} \\
& + i \sum_{h=1}^{q} \bar{\chi}(h) \left(\cot \frac{\pi h}{k} - \frac{1}{\pi h/k} \right) \\
& - \frac{ik}{\pi} \sum_{h=q+1}^{\infty} \frac{\bar{\chi}(h)}{h} = R_1 + R_2 + R_3,
\end{aligned}
$$

say. If we put $S_1(h) = \sum_{n=q+1}^{h} \bar{\chi}(n)$ and use the fact that $|S_1(h)| < 2^{-}k$, we have

$$
\begin{aligned}
(9)\qquad |R_3| &= \left| \frac{k}{\pi} \sum_{h=q+1}^{\infty} \frac{S_1(h)}{h(h+1)} \right| \leqq \frac{2^{-1}k^2}{\pi} \sum_{h=q+1}^{\infty} \frac{1}{h(h+1)} \\
&= \frac{2^{-1}k^2}{\pi(q+1)} \leqq \frac{k}{\pi}.
\end{aligned}
$$

Using the fact that $0 < x^{-1} - \cot x \leqq 2/\pi$ for $0 < x \leqq \pi/2$, we find

$$
(10)\qquad |R_2| \leqq \frac{2}{\pi} q \leqq \frac{1}{\pi} k.
$$

We divide the summation in R_1 into two parts, namely $1 \leqq h \leqq [\alpha^{-1}+1]$ and $[\alpha^{-1}+2] \leqq h \leqq q$. In the first part we use $\sin (2\pi hn/k) \leqq 2\pi hn/k$, in the second part we use $\sin (2\pi hn/k) \leqq 1$, and in both parts we use $\sin (\pi h/k) \geqq 2h/k$. Thus

$$
\begin{aligned}
|R_1| &\leqq \frac{1}{2n} \sum_{h=1}^{[\alpha^{-1}+1]} \frac{2\pi hn/k}{4h^2/k^2} + \frac{1}{2n} \sum_{h=[\alpha^{-1}+2]}^{q} \frac{1}{4h^2/k^2} \\
&\leqq \frac{\pi}{4} k \sum_{h=1}^{[\alpha^{-1}+1]} \frac{1}{h} + \frac{k^2}{8n} \sum_{h=[\alpha^{-1}+2]}^{\infty} \frac{1}{h^2} \\
(11)\qquad &\leqq \frac{\pi}{4} k \left(\frac{3}{2} + \log \frac{1}{\alpha} \right) + \frac{k^2}{8n} \alpha \\
&\leqq \left(\frac{3\pi}{8} + \frac{\pi}{4} \log \frac{1}{\alpha} + \frac{1}{8} \right) k.
\end{aligned}
$$

Combining (8), (9), (10), and (11), we get our theorem.

4. **A theorem of Davenport.** Davenport [3] has proved that if s is a fixed complex number with $0 < \sigma = R(s) < 1$, then for any primitive χ we have $|L(s, \chi)| \leqq Ck^{(1-\sigma)/2}$ where C is a constant depending on s. In this section we show how Hua's inequality (7) gives a very simple proof of this result in the case $\chi(-1) = 1$, with a specific value of the

constant. Unfortunately our Theorem 2 is too weak to give Davenport's result for the case $\chi(-1)=-1$.

THEOREM 3. *If* χ *is a primitive character with* $\chi(-1)=1$ *and if* $0<\sigma=R(s)<1$, *then*

$$|L(s,\chi)| \leqq \frac{|s(s+1)|}{\sigma(1-\sigma)} k^{(1-\sigma)/2}.$$

PROOF. Put $T(n)=\sum_{m=0}^{n} S(m)$. Then we get by two partial summations

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \sum_{n=1}^{\infty} S(n) s \int_0^1 \frac{dy}{(n+y)^{s+1}}$$
$$= \sum_{n=1}^{\infty} T(n) s(s+1) \int_0^1 \int_0^1 \frac{dxdy}{(n+x+y)^{s+2}}.$$

For $n \leqq [k^{1/2}]$ clearly $|T(n)| \leqq \sum_{m=0}^{n} m \leqq n^2$, while for $n \geqq [k^{1/2}+1]$ we have by (7)

$$|T(n)| \leqq 2^{-1}k^{1/2}\{n+1-(n+1)^2/k\} \leqq 2^{-1}k^{1/2}n.$$

Thus

$$|L(s,\chi)| \leqq |s(s+1)| \sum_{n=1}^{\infty} \frac{|T(n)|}{n^{\sigma+2}}$$
$$= |s(s+1)| \left\{ \sum_{n=1}^{[k^{1/2}]} \frac{1}{n^{\sigma}} + \sum_{n=[k^{1/2}]+1}^{\infty} \frac{2^{-1}k^{1/2}}{n^{\sigma+1}} \right\}$$
$$\leqq |s(s+1)| \left\{ 1 + \int_1^{[k^{1/2}]} \frac{dx}{x^{\sigma}} + 2^{-1}k^{1/2} \int_{[k^{1/2}]}^{\infty} \frac{dx}{x^{\sigma+1}} \right\}$$
$$\leqq |s(s+1)| \left\{ 1 + \frac{k^{(1-\sigma)/2}-1}{1-\sigma} + \frac{2^{-1}k^{1/2}}{\sigma[k^{1/2}]^{\sigma}} \right\}$$
$$\leqq |s(s+1)| \left\{ \frac{k^{(1-\sigma)/2}}{1-\sigma} + \frac{k^{(1-\sigma)/2}}{\sigma} \right\} = \frac{|s(s+1)|}{\sigma(1-\sigma)} k^{(1-\sigma)/2},$$

which proves our theorem.

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