ON CERTAIN SPECIAL SETS OF ORTHOGONAL POLYNOMIALS

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In three recent notes in C. R. Acad. Sci. Paris (vol. 228 (1949) pp. 1363–1365, 1553–1556, 1998–2000) Mr. F. Pollaczek has introduced certain sets of orthogonal polynomials with remarkable properties. In the present paper we are considering primarily the polynomials introduced in the first note, the others being certain generalizations. In view of their relatively simple definition, these polynomials can readily be studied. They serve as illustrations for certain "irregular" phenomena in the theory of orthogonal polynomials.

Extensions of the well known integral representations (integrals of Laplace and Mehler) and of other formal properties will be discussed at a later opportunity.

1. Definitions. Theorems

1. The polynomials $P_n(x; a, b)$ introduced in the first note of Mr. F. Pollaczek depend on two real parameters $a, b, a \geqq |b|$. They can be defined by their generating series

$$(1)\quad \begin{aligned} f(x, z) = f(\cos\theta, z) &= \sum_{n=0}^{\infty} P_n(x; a, b) z^n \\ &= (1 - ze^{i\theta})^{-1/2+i\phi(\theta)}(1 - ze^{-i\theta})^{-1/2-i\phi(\theta)} \end{aligned}$$

where

$$(2)\quad \phi(\theta) = \frac{a\cos\theta + b}{2\sin\theta}.$$

The alternate form

$$(3)\quad \begin{aligned} f(x, z) &= (1 - 2xz + z^2)^{-1/2} \exp\left\{ i\phi(\theta) \log \frac{1 - ze^{i\theta}}{1 - ze^{-i\theta}} \right\} \\ &= (1 - 2xz + z^2)^{-1/2} \exp\left\{ (ax + b) \sum_{m=1}^{\infty} \frac{z^m}{m} U_{m-1}(x) \right\}, \\ U_{m-1}(\cos\theta) &= \frac{\sin m\theta}{\sin\theta}, \end{aligned}$$

shows that $P_n(x; a, b)$ is indeed a polynomial of degree n. For $a = b = 0$ we obtain the Legendre polynomials.

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The basic property of these polynomials is the following orthogonality relation

$$\int_{-1}^{+1} P_n(x;\, a,\, b)P_m(x;\, a,\, b)w(x;\, a,\, b)dx = \frac{\delta_{nm}}{n + (a+1)/2} \tag{4}$$

where the weight-function is defined by

$$w(\cos\theta;\, a,\, b) = \frac{e^{(2\theta-\pi)\phi(\theta)}}{\cosh(\pi\phi(\theta))}\cdot \tag{5}$$

In §2 we give a proof of the relation (4) which is simpler than that of Mr. F. Pollaczek. It is based (instead of complex integration) on the use of Laplace transforms.

2. We have the identity

$$w(-x;\, a,\, b) = w(x;\, a,\, -b). \tag{6}$$

Let $a > |b|$. Obviously, $w(x;\, a,\, b)$ becomes zero in exponential manner at the end points $x = -1$ and $x = 1$ of the interval of orthogonality. More precisely, $x = \cos\theta$,

$$w(x;\, a,\, b) \cong 2\exp\{(a+b)(1-\pi/\theta\} \qquad \text{as } \theta \to 0 \tag{7}$$

so that log w *is not integrable in* $0 < \theta < \pi$.

3. The Jacobi polynomials associated with the weight

$$w(x) = (1-x)^\alpha(1+x)^\beta \tag{8}$$

offer a good indication how the behavior of the weight-function in the vicinity of the end points of the basic interval affects certain properties of the associated orthogonal polynomials. Let $\{p_n(x)\}$ be the *normalized* Jacobi polynomials, that is,

$$\int_{-1}^{+1} (p_n(x))^2 w(x)dx = 1. \tag{9}$$

Then[1]

(a) $p_n(1) \sim n^{\alpha+1/2}$,

(b) $p_n(x) \sim (x + (x^2-1)^{1/2})^n$,

(c) $p_n(\cos\theta) = A(\theta)\ \cos(n\theta + B(\theta)) + \epsilon_n(\theta)$, $\lim\ \epsilon_n(\theta) = 0$, $0 < \theta < \pi$.

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[1] Cf. G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23, 1939, (4.1.1), p. 57, (8.21.9) and (8.21.10), p. 190, (4.3.3), p. 67. In the following we refer to this book as OP.

The sign $\sim$ indicates that the ratio of the given quantities approaches a limit different from 0. (We use the sign $\cong$ if this limit is 1.) The variable x is confined to the complex plane cut along the segment -1, 1 and $(x^2-1)^{1/2}>0$ for $x>1$. The functions $A(\theta)$, $B(\theta)$ are independent of n and analytic in $0<\theta<\pi$, $A(\theta)\neq 0$.

4. It is interesting to compare (a)–(c) with the corresponding properties of the orthogonal polynomials $P_n(x;\, a,\, b)$ defined above. Let $a>|b|$; we write

$$(n + (a + 1)/2)^{1/2}P_n(x;\, a,\, b) = p_n(x;\, a,\, b) \tag{10}$$

so that $\{p_n(x;\, a,\, b)\}$ is the orthonormal set associated with the weight-function $w(x;\, a,\, b)$. Now,

$$P_n(1;\, a,\, b) = L_n(-a - b) \tag{11}$$

where L_n is the Laguerre polynomial so that (OP, Theorem 8.22.3, p. 193):

(a′) $p_n(1;\, a,\, b)\sim n^{1/4} \exp \{2(a+b)^{1/2}n^{1/2}\}$,

(b′) $p_n(x;a,b)\sim n^K(x+(x^2-1)^{1/2})^n, K=K(x)=(ax+b)/2(x^2-1)^{1/2}$,

(c′) $p_n(\cos \theta;\, a,\, b)=A_1(\theta) \cos (n\theta-\phi(\theta) \log n+B_1(\theta))+\epsilon_n(\theta)$

where $(x^2-1)^{1/2}>0$ for $x>1$ and $A_1(\theta)$, $B_1(\theta)$, $\epsilon_n(\theta)$ are functions of the same kind as in (c).

We observe the difference in the asymptotic behavior displayed by the formulas (a), (a′), (b), (b′), and (c), (c′). According to OP, Theorem 12.1.2 (p. 290) a formula of the type (b) holds whenever $w(x)$ is an arbitrary weight-function such that $\log w(\cos \theta)$ is integrable in 0, π. According to 2. this is not the case for our weight-function $w(x;\, a,\, b)$.

As mentioned, the subject of §2 is a new proof of the orthogonality relation (4). In §3 we discuss the asymptotic relations (b′), (c′). §4 deals with a class $P_n^{(\lambda)}(x;\, a,\, b)$ of polynomials related to the ultraspherical polynomials $P_n^{(\lambda)}(x)$ (OP, 4.7, p. 80) in the same way as the polynomials $P_n(x;\, a,\, b)$ are related to the Legendre polynomials.

2. Proof of the orthogonality relation

Let z_1 and z_2 be real, $|z_1|<1$, $|z_2|<1$. We have

$$(1 - z_1e^{i\theta})(1 - z_2e^{i\theta}) = e^{i(\theta-\pi/2)}(1 - z_1z_2) \sin \theta(1 + iH),$$

$$H = \frac{(1 + z_1z_2) \cos \theta - (z_1 + z_2)}{(1 - z_1z_2) \sin \theta}, \tag{1}$$

so that, $x=\cos \theta$,

$$
\begin{aligned}
&f(x,z_1)f(x,z_2)w(x;\, a,\, b)\sin\theta \\
&\quad = e^{i(\theta-\pi/2)\cdot 2i\phi(\theta)}((1-z_1z_2)\sin\theta)^{-1}(1+iH)^{-1/2+i\phi(\theta)} \\
(2)\qquad &\qquad\qquad\qquad \cdot(1-iH)^{-1/2-i\phi(\theta)}w(x;\, a,\, b)\sin\theta \\
&\quad = (\cosh(\pi\phi(\theta)))^{-1}(1-z_1z_2)^{-1}(1+iH)^{-1/2+i\phi(\theta)}(1-iH)^{-1/2-i\phi(\theta)}.
\end{aligned}
$$

Using the familiar formula

$$
(3)\qquad m^{-\alpha} = (\Gamma(\alpha))^{-1}\int_0^\infty e^{-ms}s^{\alpha-1}ds, \qquad \Re m > 0,\, \Re\alpha > 0,
$$

we obtain for the integral I of (2), $0\leqq\theta\leqq\pi$,

$$
\begin{aligned}
I &= (1-z_1z_2)^{-1}\int_0^\pi \{\cosh(\pi\phi(\theta))\Gamma(1/2-i\phi(\theta))\Gamma(1/2+i\phi(\theta))\}^{-1} \\
&\qquad \cdot\int_0^\infty\int_0^\infty e^{-(1+iH)s_1-(1-iH)s_2}s_1^{-1/2-i\phi(\theta)}s_2^{-1/2+i\phi(\theta)}ds_1ds_2\cdot d\theta \\
(4)\qquad & = \pi^{-1}(1-z_1z_2)^{-1}\int_0^\infty\int_0^\infty e^{-s_1-s_2}(s_1s_2)^{-1/2} \\
&\qquad \cdot\int_0^\pi e^{-iH(s_1-s_2)-i\phi(\theta)(\log s_1-\log s_2)}d\theta\cdot ds_1ds_2.
\end{aligned}
$$

Interchanging s_1 and s_2 or replacing θ by $-\theta$ has the same effect on the last integrand so that

$$
\begin{aligned}
I &= \pi^{-1}(1-z_1z_2)^{-1}\iint_{s_1\geqq s_2} e^{-s_1-s_2}(s_1s_2)^{-1/2} \\
(5)\qquad &\qquad \cdot\int_{-\pi}^{\pi} e^{-iH(s_1-s_2)-i\phi(\theta)(\log s_1-\log s_2)}d\theta\cdot ds_1ds_2.
\end{aligned}
$$

Now[2]

$$
\begin{aligned}
&\int_{-\pi}^{\pi}\exp\left(-i\,\frac{A\cos\theta+B}{\sin\theta}\right)d\theta \\
(6)\qquad &\qquad = \frac{1}{i}\int \exp\left(\frac{A(\zeta^2+1)+2B\zeta}{\zeta^2-1}\right)\zeta^{-1}d\zeta = 2\pi e^{-A}
\end{aligned}
$$

where the complex integration has to be extended over $|\zeta|=1$ with *indentations* around $\zeta=\pm1$. The contribution of the indentations vanishes provided $A\pm B\geqq 0$ which is indeed the case since

[2] This already proves the orthogonality since A depends only on z_1z_2.

$$
(7)\qquad
\begin{aligned}
A &= \frac{1+z_1z_2}{1-z_1z_2}(s_1-s_2)+\frac{a}{2}(\log s_1-\log s_2),\\
B &= -\frac{z_1+z_2}{1-z_1z_2}(s_1-s_2)+\frac{b}{2}(\log s_1-\log s_2).
\end{aligned}
$$

Hence, $s_1=s_2e^\sigma$,

$$
(8)\qquad
\begin{aligned}
I &= 2(1-z_1z_2)^{-1}\iint_{s_1\geqq s_2}\\
&\quad\cdot\exp\left\{-s_1-s_2-\frac{1+z_1z_2}{1-z_1z_2}(s_1-s_2)\right\}s_1^{-(1+a)/2}s_2^{-(1-a)/2}ds_1ds_2\\
&= 2(1-z_1z_2)^{-1}\int_0^\infty\int_0^\infty\\
&\quad\cdot\exp\left\{-s_2(e^\sigma+1)-\frac{1+z_1z_2}{1-z_1z_2}s_2(e^\sigma-1)\right\}e^{-\sigma(1+a)/2}e^\sigma ds_2d\sigma\\
&= 2(1-z_1z_2)^{-1}\int_0^\infty\left\{e^\sigma+1+\frac{1+z_1z_2}{1-z_1z_2}(e^\sigma-1)\right\}^{-1}e^{-\sigma(1+a)/2}e^\sigma d\sigma\\
&= \int_0^\infty\frac{e^{-\sigma(1+a)/2}}{1-e^{-\sigma}z_1z_2}d\sigma=\sum_{n=0}^\infty\frac{z_1^nz_2^n}{n+(a+1)/2}
\end{aligned}
$$

which proves §1 (4).

3. Asymptotic formulas

1. The generating function $f(x, z)$ in §1 (1) has two distinct singular points $e^{\pm i\theta}$ unless $-1/2+i\phi(\theta)$ is an integer (or infinite, that is, $\theta=k\pi$, k integer).

Let $\Im\theta>0$. The next singular point to the origin is $e^{i\theta}=x-(x^2-1)^{1/2}$. Applying Darboux's method we obtain

$$
(1)\qquad P_n(x;a,b)\cong(1-e^{2i\theta})^{-1/2+i\phi(\theta)}C_{-1/2-i\phi(\theta),n}(-e^{-i\theta})^n.
$$

Since

$$
(2)\qquad
\begin{aligned}
C_{-1/2-i\phi(\theta),n} &= (-1)^n\frac{\Gamma(n+1/2+i\phi(\theta))}{\Gamma(n+1)\Gamma(1/2+i\phi(\theta))}\\
&\cong\frac{(-1)^nn^{-1/2+i\phi(\theta)}}{\Gamma(1/2+i\phi(\theta))},
\end{aligned}
$$

we find for the right-hand expression in (1)

$$(3) \qquad n^{-1/2}\cdot\frac{(1-e^{2i\theta})^{-1/2+i\phi(\theta)}}{\Gamma(1/2+i\phi(\theta))}\cdot n^{i\phi(\theta)}e^{-in\theta}.$$

This establishes §1 (b′).

The result holds also if $-1/2+i\phi(\theta)$ is an integer equal to m. This integer must be necessarily non-negative so that $f(x, z)$ has a single pole of order $m+1$ at $e^{i\theta}$. Indeed, let $\zeta=e^{i\theta}$, $|\zeta|<1$; we have

$$(a+2m+1)\zeta^2+2b\zeta+(a-2m-1)=0.$$

(We have $a+2m+1\neq 0$, otherwise $\zeta=-a/b$, $|\zeta|\geqq 1$.) If we assume that ζ is not real, the other root of this equation would be $\bar{\zeta}$ so that

$$|\zeta\bar{\zeta}|=\left|\frac{a-2m-1}{a+2m+1}\right|<1, m\geqq 0.$$

In case $\theta=it$ or $it+\pi$, $t>0$, it is easy to see that $-1/2+i\phi(\theta)\geqq -1/2$.

2. Let $0<\theta<\pi$. Applying Darboux's method again we find

$$\begin{aligned}&P_n(\cos\theta; a, b)\\ &=2\Re(1-e^{2i\theta})^{-1/2+i\phi(\theta)}C_{-1/2-i\phi(\theta),n}(-e^{-i\theta})^n+O(n^{-3/2})\\ &=2n^{-1/2}\Re\frac{\exp\{(-1/2+i\phi(\theta))(\log(2\sin\theta)+i(\theta-\pi/2))-in\theta+i\phi(\theta)\log n\}}{\Gamma(1/2+i\phi(\theta))}\\ &\quad+O(n^{-3/2}).\end{aligned}$$

This establishes §1 (c′).

4. A generalization

1. Let $\lambda>-1$, a and b real, $a\geqq|b|$. We introduce the weight-function

$$(1)\quad w^{(\lambda)}(x; a, b)=\pi^{-1}2^{2\lambda-1}e^{(2\theta-\pi)\phi(\theta)}|\Gamma(\lambda+i\phi(\theta))|^2(1-x^2)^{\lambda-1/2}$$

where $x=\cos\theta$ and $\phi(\theta)$ has the same meaning as in §1 (2). The generating function

$$(2)\qquad \begin{aligned} f^{(\lambda)}(x, z) &= \sum_{n=0}^{\infty} P_n^{(\lambda)}(x; a, b)z^n\\ &=(1-ze^{i\theta})^{-\lambda+i\phi(\theta)}(1-ze^{-i\theta})^{-\lambda-i\phi(\theta)}\end{aligned}$$

defines certain polynomials $P_n^{(\lambda)}(x; a, b)$. We have $P_n^{(1/2)}(x; a, b)=P_n(x; a, b)$ and $P_n^{(\lambda)}(x; a, b)$ reduces to the ultraspherical polynomials for $a=b=0$.

We prove the orthogonality relation

$$\int_{-1}^{+1} P_n^{(\lambda)}(x; a, b) P_m^{(\lambda)}(x; a, b) w^{(\lambda)}(x; a, b) dx = \frac{\Gamma(n+2\lambda)}{n!} \frac{\delta_{nm}}{n+\lambda+a/2}. \tag{3}$$

2. The proof follows a similar line as in §2 so that we can be brief. We have

$$\begin{aligned} f^{(\lambda)}(x, z_1) f^{(\lambda)}(x, z_2) w^{(\lambda)}(x; a, b) \sin\theta &= \pi^{-1} 2^{2\lambda-1}(1 - z_1 z_2)^{-2\lambda} \\ &\cdot |\Gamma(\lambda + i\phi(\theta))|^2 (1 + iH)^{-\lambda+i\phi(\theta)} (1 - iH)^{-\lambda-i\phi(\theta)} \end{aligned} \tag{4}$$

so that we obtain for the integral $I^{(\lambda)}$ of this expression:

$$\begin{aligned} I^{(\lambda)} &= \pi^{-1} 2^{2\lambda-1}(1 - z_1 z_2)^{-2\lambda} \\ &\cdot \int_0^{\pi} \int_0^{\infty} \int_0^{\infty} e^{-(1+iH)s_1 - (1-iH)s_2} s_1^{\lambda-1-i\phi(\theta)} s_2^{\lambda-1+i\phi(\theta)} ds_1 ds_2 \cdot d\theta. \end{aligned} \tag{5}$$

As in the case $\lambda = 1/2$ we are led to

$$\begin{aligned} I^{(\lambda)} &= 2^{2\lambda}(1 - z_1 z_2)^{-2\lambda} \iint_{s_1 \geqq s_2} \exp\left\{ - s_1 - s_2 - \frac{1 + z_1 z_2}{1 - z_1 z_2}(s_1 - s_2) \right\} \\ &\qquad \cdot s_2^{\lambda-1-a/2} s_2^{(\lambda-1+a/2)} ds_1 ds_2 \\ &= 2^{2\lambda}(1 - z_1 z_2)^{-2\lambda} \int_0^{\infty} \int_0^{\infty} \exp\left\{ -s_2(e^{\sigma} + 1) - \frac{1 + z_1 z_2}{1 - z_1 z_2} s_2(e^{\sigma} - 1) \right\} \\ &\qquad \cdot s_2^{2\lambda-1} e^{\sigma(\lambda-1-a/2)} e^{\sigma} ds_2 d\sigma \\ &= 2^{2\lambda}(1 - z_1 z_2)^{-2\lambda} \Gamma(2\lambda) \int_0^{\infty} \left\{ e^{\sigma} + 1 + \frac{1 + z_1 z_2}{1 - z_1 z_2}(e^{\sigma} - 1) \right\}^{-2\lambda} \\ &\qquad \cdot e^{\sigma(\lambda-1-a/2)} e^{\sigma} d\sigma \\ &= \Gamma(2\lambda) \int_0^{\infty} (1 - e^{-\sigma} z_1 z_2)^{-2\lambda} e^{-\sigma(\lambda+a/2)} d\sigma \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(n + 2\lambda)}{n!} \frac{z_1^n z_2^n}{n + \lambda + a/2}. \end{aligned}$$

This establishes (3).

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