THE BOUNDED ADDITIVE OPERATION ON BANACH SPACE

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In his book on the theory of linear operations (p. 54) Banach proves that an additive operation U on a normed linear space to another is continuous (hence, linear, at least for real spaces) if and only if it satisfies the Lipschitz condition $|U(x)| < M|x|$.

Without changing the proof given there in any essential detail, one proves that an additive, homogeneous operation is continuous if and only if it is bounded on the interior, or, equivalently, the circumference of the unit sphere. We show here that the assumption of homogeneity can be dropped. This, however, requires an essentially different argument. The one we give here goes back to Cauchy (*Cours d'analyse de l'École Royale Polytechnique*, part 1, *Analyse algébrique*, Paris, 1821) and Hamel (Math. Ann. vol. 60 (1905) pp. 459–562).

THEOREM. *An additive operation U on a normed linear space to another is continuous* (*hence real-linear*) *if and only if it is bounded on the interior of the unit sphere. The interior cannot be replaced by the circumference in this statement.*

The only novelty lies in the proof of sufficiency. Let $|U(x)| < M$ for $|x| < 1$. Choose an arbitrary x in the space with $|x| < 1$. For any real number y set $f(y) = U(yx) - yU(x)$. Since f is additive, $f(ry) = rf(y)$ whenever r is rational. In particular $f(r) = 0$ if r is rational since $f(1) = 0$. Moreover

$$|f(y)| < 2M \qquad \text{if} \quad 0 \leqq y < 1. \tag{1}$$

Now, for an arbitrary positive rational number r and an arbitrary real number y choose a rational number s so that $0 \leqq ry + s < 1$. Then, from (1), $|f(ry+s)| < 2M$. But $f(ry+s) = rf(y)$ and so $r|f(y)| < 2M$. Since r is arbitrary it follows that $f(y) = 0$, that is, $U(yx) = yU(x)$ if $|x| < 1$. For arbitrary x we have $U(yx) = U(2y|x|x/2|x|) = 2y|x| U(x/2|x|) = yU(x)$. Thus U is homogeneous and for arbitrary x, $U(x) = 2|x| U(x/2|x|) < 2M|x|$ from which follows the continuity as in Banach's proof.

To prove the second part of the theorem we note, for example, that in the space of real numbers all functions are bounded on the unit circumference.

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