A FIXED POINT THEOREM FOR PSEUDO-ARCS AND CERTAIN OTHER METRIC CONTINUA

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E. E. Moise has defined a pseudo-arc [1][1] and has shown that it has the property of being homeomorphic to each of its nondegenerate subcontinua. Bing has shown that a pseudo-arc is homogeneous [2]. F. B. Jones, in a letter to the author, has raised the question as to whether a pseudo-arc has the fixed point property with respect to continuous transformations. It is the purpose of this note to show that a pseudo-arc is a member of a more general class of metric continua which have the fixed point property.

We shall use Moise's definition of a chain, namely: *a chain Y is a collection of mutually exclusive open sets* (called links), $y_1, y_2, y_3, \cdots, y_k$, *such that y_i and y_j have a boundary point in common if and only if i and j are identical or consecutive integers.* If Y designates a chain, $C(Y)$ will designate the closure of the set of points each of which is in a link of the chain Y, and ΔY will designate the maximum diameter of a link of the chain Y. For convenience the closure of a link, $C(y_i)$, of Y will be called a closed link of Y. A closed link $C(y_i)$ of Y will be said to precede $C(y_j)$ in Y if $i<j$ and to follow $C(y_j)$ if $i>j$.

THEOREM. *Let $Y_1, Y_2, Y_3, \cdots$ be a sequence of chains such that:* (1) *$C(Y_1)$ is a compact nonvacuous metric space,* (2) *$C(Y_{i+1})$ is a subset of $C(Y_i)$ for each i,* (3) $\text{limit}_{i\to\infty} \Delta Y_i = 0$. *Let M designate the continuum which is the intersection of the sets $C(Y_i)$. Then if T is a continuous transformation of M into a subset of itself, there is a point p of M such that $T(p)=p$.*

PROOF. Let ϵ be a positive real number. Let m be a positive integer such that ΔY_m is less than ϵ. Let A be the subset of M consisting of all points p of M such that either each of the closed links of Y_m which contain $T(p)$ follows all the closed links of Y_m which contain p or such that $T(p)$ is contained in some closed link of Y_m which contains p. Let B be the subset of M consisting of all points p of M such that either each closed link of Y_m which contains $T(p)$ precedes all the closed links of Y_m which contain p or such that $T(p)$ is contained in some closed link of Y_m which contains p. The points of M in the first closed link of Y_m are in A and the points of M in the last

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[1] Numbers in brackets refer to the references cited at the end of the paper.

closed link of Y_m are in B, and hence these sets are nonvacuous. It is easily shown that A and B are closed. Then the compact continuum M is the sum of the two closed sets A and B, which therefore have a point q in common. Hence q and $T(q)$ lie together in some closed link z of Y_m, and the distance from q to $T(q)$ is less than ϵ. Since ϵ is an arbitrary positive number, and since T is continuous and M is closed, it follows that for some point p of M, $T(p)=p$.

COROLLARY. *A pseudo-arc has the fixed point property for continuous transformations.*

PROOF. A pseudo-arc is a point set satisfying the definition of the set M of the theorem.

It should be noted that the theorem applies to certain indecomposable continua which are not pseudo-arcs. An example is the one given by Brouwer [**3**] and cited by Urysohn [**4**].

REFERENCES

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4. Paul Urysohn, *Mémoire sur les multiplicités cantoriennes*, Fund. Math. vol. 8 (1926) pp. 225–251.

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