ARCWISE CONVEX SETS

F. A. VALENTINE

1. The concept of arcwise convex set is a natural generalization of the notion of L set. The latter was studied by Horn and Valentine [2][1]. It is especially interesting to observe how the theorem about the complement of an arcwise convex continuum sheds light on the corresponding theorem for L sets. In order to make this precise, the following definitions are used.

DEFINITION 1. *An arc C is said to be convex if it is contained in the boundary of its convex hull, denoted by $H(C)$.*

DEFINITION 2. *A set S is said to be arcwise convex if each pair of points in S can be joined by a convex arc in S.*

DEFINITION 3. *A continuum S in the Euclidean plane is called a unilaterally connected continuum if each pair of points x and y in S lies in a subcontinuum of S which is contained in one of the closed half-planes determined by the line passing through x and y.*

An L set is an arcwise convex set, since, by definition, each pair of points in L can be joined by a polygonal line in L containing at most two segments.

NOTATION 1. We let xy denote the closed linear segment joining x and y, and $L(x, y)$ denotes the straight line passing through x and y. The convex hull of C is denoted by $H(C)$. The two open half-planes determined by a line $L(x, y)$ in a plane are denoted by $R^+(x, y)$ and $R^-(x, y)$. A component of the complement of S is called K.

In this paper we restrict ourselves to sets in a two-dimensional Euclidean space E_2. The principal theorem in this section is:

THEOREM 1. *Each component of the complement of a unilaterally connected continuum*[2] *$S \subset E_2$ is an arcwise convex set.*

COROLLARY 1. *Each component of the complement of an arcwise convex continuum $S \subset E_2$ is an arcwise convex set.*

In order to prove Theorem 1 we first prove several lemmas. Let K be a component of the complement of a unilaterally connected continuum S. Since K is open, it is polygonally connected.

DEFINITION 4. *Among the polygonal arcs in K joining x and y,*

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] A continuum in E_2 is a bounded closed connected set.

there exist those that have a minimum number of segments. Each is called an irreducible P_n *in* K *(having* n *segments).*

DEFINITION 5. *A three-sided polygonal arc* P_3 *is called a z-shaped* P_3 *if* $H(P_3)-P_3$ *has two components.*

LEMMA 1. *An irreducible* $P_n \subset K$ *contains no z-shaped* P_3 *subarcs.*

PROOF. The proof is almost identical to that given by Horn and Valentine for cases 1 and 2 in Theorem 1 of [2, p. 132]. To explain briefly, if P_n did contain such a P_3 subarc, there would exist a segment ac, with $a \in P_3$, $c \in P_3$, and also points $b \in ac \cdot P_3$, $s_i \in ac \cdot S$ $(i=1, 2)$ with an order $a < s_1 < b < s_2 < c$ on ac. Moreover, the subarcs of P_3 joining a to b and b to c would lie in the different half-planes determined by $L(a, c)$. Hence no subcontinuum in S can contain s_1 and s_2 and lie in one of the half-planes determined by $L(a, c)$. But this contradicts the fact that S is a unilaterally connected continuum.

LEMMA 2. *If* $P_n \subset K$ *is an irreducible polygonal arc joining* x *and* y, *and if* P_n *lies in one of the closed half-planes determined by* $L(x, y)$, *then* P_n *is a convex arc.*

PROOF. Each angle in the polygon consisting of P_n and xy is less than π, since P_n contains no z-shaped subarcs.

LEMMA 3. *If* $P_n \subset K$ *is irreducible* $(n>1)$, *then the boundary of* $H(P_n)$ *consists of a convex subarc of* P_n *plus an open segment in the complement of* P_n.

PROOF. Since the boundary of $H(P_n)$, denoted by $B(H)$, is not P_n, there exist two points $\alpha \in B(H) \cdot P_n$, $\beta \in B(H) \cdot P_n$ such that $(\alpha\beta - \alpha - \beta) \cdot P_n = 0$, and such that $L(\alpha, \beta)$ is a line of support to $H(P_n)$. Hence α and β are joined by a subarc P_r of P_n which lies on one side of $L(\alpha, \beta)$. Thus, by Lemma 2, P_r is a convex arc. Moreover, the set $P_n - P_r$ is contained in the interior of $H(P_r)$.

NOTATION 2. *If* $\xi \in B(H) \cdot P_n$, $\eta \in B(H) \cdot P_n$, the convex subarc of P_n joining ξ and η is denoted by $P(\xi, \eta)$.

LEMMA 4. *Suppose that* K *is a bounded component of the complement of a unilaterally connected continuum. Let* x_i $(i=1, 2, 3)$ *be three collinear points on a polygonal arc* $Q \subset K$. *Suppose the two subarcs of* Q *joining* x_1 *to* x_2 *and* x_1 *to* x_3, *respectively, lie on opposite sides of* $L(x_1, x_3)$. *Then either* x_1x_2 *or* x_1x_3 *is in* K.

PROOF. Without loss of generality first assume x_2 is between x_1 and x_3. Since K is bounded, there exists a point $s \in [L(x_1, x_3) - x_1x_3] \cdot S$. If $x_1x_2 \cdot S \neq 0$, any subcontinuum of S joining s to a point $s_1 \in x_1x_2 \cdot S$

and lying on one side of $L(x_1, x_3)$ would intersect Q. Hence $x_1x_2 \subset K$. Similarly, if $x_1 \in x_2x_3$, then either x_1x_2 or x_1x_3 is in K.

Proof of Theorem 1. I. First, let K be a bounded component of the complement of S. Choose $x \in K$, $y \in K$. If x and y are in the boundary of $H(P_n)$, then $P_n = P(x, y)$ is convex (see Notation 2). Hence suppose x is interior to $H(P_n)$. Without loss of generality suppose in going from x to y on P_n we have in order x, α, β, y $(x \neq \alpha)$, where $\alpha\beta$ is the segment described in Lemma 3. We prove that $\alpha x \subset K$, $\beta y \subset K$. Suppose $\alpha x \not\subset K$. Let $\beta \in R^+(\alpha, x)$ (see Notation 1). Also let $L(\alpha, x) \cdot H(P_n) \equiv \alpha x_1$, so that $x_1 \in P_n$. The convex subarc of P_n joining α and x_1 is such that $P(\alpha, x_1) \subset R^-(\alpha, x)$. Since, by Lemma 1, P_n contains no z-shaped subarcs, a subarc P_s of P_n exists joining α to a point τ, where $\alpha x \subset \alpha\tau \subset \alpha x_1$, and $P_s \subset R^+(\alpha, x)$. This together with Lemma 4 implies that $\alpha\tau \subset K$. Since $\alpha x \subset \alpha\tau$, we have $\alpha x \subset K$. In exactly the same way $\beta y \subset K$.

(1) If $L(x, y) \cdot \alpha\beta = 0$, let $L(x, y) \cdot B[H] \equiv \xi + \eta$, where $\xi \in R^-(\alpha, x)$, $\eta \in R^+(\alpha, x)$. Since $\alpha x \subset K$, $\beta y \subset K$, Lemma 4 implies that $\xi x \subset K$, $\eta y \subset K$. If x is between ξ and y, then $x\xi + P(\xi, \eta) + \eta y$ is a convex arc in K joining x and y. If y is between x and ξ, then $y\xi + P(\xi, \eta) + \eta x$ is the desired convex arc.

(2) If $L(x, y) \cdot \alpha\beta \neq 0$, let $L(x, y) \cdot (P_n - \alpha - \beta) \equiv \xi$. If y is between x and ξ, then since $y\beta + P(\beta, \xi) \subset \overline{R}^+(x, y)$, and since $x\alpha + P(\alpha, \xi) \subset \overline{R}^-(x, y)$, Lemma 4 implies that $y\xi \subset K$. Hence $y\xi + P(\xi, \alpha) + \alpha x$ is the desired convex arc in K. Similarly if x is between y and ξ, $x\xi + P(\beta, \xi) + \beta y$ is the desired convex arc. This completes the proof when K is bounded.

II. Second, let K be the unbounded component of the complement of S. We shall prove that through each point $x \in K$ there passes a half-line in K having x as end point. Since S is closed and bounded, there exists an irreducible polygonal ray $P_n \subset K$ joining x to ∞ (n is finite). Let $P_n \equiv x\ x_1\ x_2 \cdots x_{n-1}\infty$, where x_i and x_{i+1} are consecutive vertices. Assume no half-line in K exists having x as end point. Then $n \geqq 1$, so that the segment x_1x_2 may be considered as a finite segment or as an infinite half-line.

(1) Suppose that x_1x_2 is a finite segment. Let $L(\theta)$ be the half-line which has x as end point, which intersects x_1x_2, and which makes an angle θ with xx_1 $(0 \leqq \theta < \pi)$. Let ϕ be the angle such that $xx_2 \subset L(\phi)$. Hence $0 < \phi < \pi$. Let D be the closed 2-dimensional triangle determined by x, x_1, x_2, and define $C \equiv \overline{E_2 - D}$. Since $xx_1 \cdot S = 0$, we must have $L(0) \cdot S \cdot C \neq 0$. Since P_n is irreducible, $xx_2 \cdot S \neq 0$, so that $L(\phi) \cdot S \cdot D \neq 0$. By considering the two sets of angles from the set $0 \leqq \theta \leqq \phi$ for which $L(\theta) \cdot S \cdot D \neq 0$ or for which $L(\theta) \cdot S \cdot C \neq 0$,

each is a closed nonempty set since S is a continuum. Hence there exists an angle w $(0<w\leqq\phi)$ such that $L(w)\cdot S\cdot C\neq 0$, $L(w)\cdot S\cdot D\neq 0$. Define $L(w)\cdot x_1x_2=t$, where t must be a point. Choose $s_1\in L(w)\cdot S\cdot D$, $s_2\in L(w)\cdot S\cdot C$. We have $s_1\neq s_2$, since $s_1\neq t$. Let $S_1\subset S$ be a subcontinuum joining s_1 and s_2 and lying on one side of $L(x, t)$. Since there exists a half-plane $R^+(x, t)$ such that $xx_1+x_1t\subset\overline{R}^+(x, t)$, and since S is a unilaterally connected continuum, we have $P_{n-1}\equiv(tx_2x_3\cdots x_{n-1}\infty-x_2)\subset R^-(x, t)$. Hence if $S_1\subset\overline{R}^+(x, t)$, then $S_1\cdot(xx_1+x_1t)\neq 0$, whereas if $S_1\subset\overline{R}^-(x, t)$, then $S_1\cdot P_{n-1}\neq 0$. But this is a contradiction, since $P_n\cdot S=0$, and since S is a unilaterally connected continuum. Hence our assumption that no half-line through x exists in K is false in this case.

(2) If $P_n=xx_1\infty$, the proof is the same as above provided that we identify x_1x_2 with $x_1\infty$, and if we let $x\infty$ be a half-line through x parallel to $x_1\infty$ such that $x\infty+xx_1+x_1\infty$ is a convex unbounded polygon D. By letting ϕ be the angle between $x\infty$ and xx_1, the proof is the same as in (1).

Finally to show that $x\in K$ and $y\in K$ (K being unbounded) can be joined by a convex arc in K, set up an order on $L(x, y)\equiv L$ from x to y so that $x<y$. We shall prove that there exist two half-rays $L(x)$ and $L(y)$ having x and y as end points, respectively, both of which are in $\overline{R}^+(x, y)\cdot K$ or in $\overline{R}^-(x, y)\cdot K$. If $L(x)$ exists such that $L(x)\subset L\cdot K$, then any $L(y)\subset K$ will do. Hence suppose that $S\cdot(-\infty x)\neq 0$, $S\cdot(y\infty)\neq 0$, where $-\infty x$ and $y\infty$ are the two components of $L-xy$. We know that $L(x)$ and $L(y)$ exist in K. Without loss of generality assume that $L(x)-x\subset R^+(x, y)$, $L(y)-y\subset R^-(x, y)$. Then any continuum $S_1\subset S$ which joins $s_1\in(-\infty x\cdot S)$ to a point $s_2\in(S\cdot y\infty)$, and which lies in $\overline{R}^+(x, y)$ or in $\overline{R}^-(x, y)$, must intersect $L(x)$ or $L(y)$ respectively, since $s_1<x<y<s_2$. This is a contradiction, since S is a unilaterally connected continuum. Hence two half-lines $L(x)$ and $L(y)$ in K exist, which, by an appropriate relabeling, lie in $\overline{R}^+(x, y)$. Since K is open, and since S is bounded, one of these rays may be rotated in $\overline{R}^+(x, y)$ so that $L(x)$ and $L(y)$ are not both in $L(x, y)$. Since S is bounded, one may choose $x_1\in L(x)$, $y_1\in L(y)$ so that $x_1y_1\cdot S=0$. The points x_1 and y_1 may or may not be distinct. They may be chosen so that the polygonal arc $xx_1+x_1y_1+y_1y$ is a convex arc in K. This completes the proof.

The above proof shows the following to be true.

THEOREM 2. *Let K be the unbounded component of the complement of a unilaterally connected continuum. Each pair of points in K can be joined by a convex three-sided polygonal line lying in K.*

2. In this section we investigate a generalization of the notion of Kerneigebiet developed by Brunn [1]. See also Valentine [3].

DEFINITION 6. *Let x be a point such that every point of a set S can be joined to x by a convex arc lying in S. Designate the set of all such points x by $K(S)$.*

DEFINITION 7. *Let $S \subset E_2$. A convex arc $C \subset S$ joining x and y is said to be a minimal convex arc in S provided for any other convex arc $C_1 \subset S$ joining x and y, either $H(C_1) \supset H(C)$ or $H(C_1) \cdot H(C) \subset L(x, y)$.*

THEOREM 3. *If S is a simply connected closed set in E_2, then $K(S)$ is an arcwise convex set.*

PROOF. Since S is simply connected, each pair of points in $K(S)$ is joined by a unique minimal convex arc. Such a minimal arc in S must exist for each pair of points x and y in $K(S)$, since S is closed and since at least one convex arc in S joins x and y. Choose $x_j \in K(S)$ $(j=1, 2)$, and let A_3 be the minimal convex arc in S joining x_1 and x_2. We shall prove that $K(S)$ is arcwise convex, by showing that $A_3 \subset K(S)$. To do this choose any point $x_3 \in S$, and let A_1 and A_2 be the minimal convex arcs in S joining x_3 to x_2 and x_1 respectively. First, we show that $A_i \cdot A_j$ $(i \neq j;\ i, j=1, 2, 3)$ is a convex arc. In the future we understand that the subscripts i, j, k satisfy the conditions $(i \neq j \neq k \neq i;\ i, j, k=1, 2, 3)$ unless otherwise stated. Suppose, for instance, $A_2 \cdot A_3$ were not a convex arc. Then there would exist a point $a_1 \in A_2 \cdot A_3$ with $a_1 \neq x_1$, such that the subarcs of A_2 and A_3 joining x_1 to a_1 would not coincide. But since S is simply connected, it is easy to prove that this implies that either A_2 or A_3 would not be a minimal convex arc. Hence $A_i \cdot A_j$ are convex arcs. They may be points. (A point is also a convex arc.) If $A_i \cdot A_j$ is not a point, let its end point, other than x_k, be a_k; otherwise let $a_k = x_k$. If $A_3 \subset A_1 + A_2$, then any point $x \in A_3$ can be joined to x_3 by a convex subarc of A_1 or A_2. Hence consider the situation when $A_3 \not\subset (A_1 + A_2)$, so that setting up an order on A_3 from x_1 to x_2 we have $x_1 \leqq a_1 < a_2 \leqq x_2$. Hence the arcs $C_i \equiv \overline{A_i - A_i \cdot (A_j + A_k)}$ are such that $C_i \cdot C_j = a_k$ with $a_j \neq a_k$. Hence the simple closed arc $\sum_{i=1}^{3} C_i$ bounds the closure R of a simply connected set. Also $R \subset S$, since $\sum_{i=1}^{3} C_i \subset S$, and since S is simply connected. Each curve C_i is a *concave* edge of R, that is, $H(C_i) \cdot R = C_i$. An equivalent statement is that $C_i \subset \Delta(a_1, a_2, a_3)$, where $\Delta(a_1, a_2, a_3)$ is the triangle determined by a_i $(i=1, 2, 3)$. To prove this, suppose, for instance, that $H(C_1) \cdot R \neq C_1$. Since $C_1 \neq a_2a_3$, a line of support λ to $H(C_1)$ exists which is parallel to a_2a_3, and which does not contain a_2a_3. Choose a point $y \in \lambda \cdot H(C_1)$. There must exist a sequence of interior points y_t $(t=1, 2, \cdots)$ of $H(C_1) \cdot R$

which has y as a limit point. Let T_t be the chord of C_1 which passes through y_t, and which is parallel to λ. Since $(C_1-a_2-a_3)\cdot(C_2+C_3)=0$, there must exist a chord T_r such that $T_r\cdot(C_2+C_3)=0$. Since $T_r\subset S\cdot H(C_1)$, it is clear that C_1, and hence A_1, would not be a minimal convex arc. Similarly $H(C_i)\cdot R=C_i$ $(i=1, 2, 3)$.

Now since any point $x\in A_3\cdot(A_1+A_2)$ can be joined to x_3 by a convex subarc of A_1 or A_2, choose $x\in[A_3-A_3\cdot(A_1+A_2)]$. We assumed that $A_3\not\subset(A_1+A_2)$. Since each arc C_i is a concave edge of R, and since $C_i\cdot C_j=a_k$, the points a_i $(i=1, 2, 3)$ cannot be collinear. The three lines $L(a_i, a_j)$ determine exactly seven open polygonal regions. Let $D(a_i)$ be the v-shaped unbounded region whose closure contains only a_i, and let $D(a_i, a_j)$ be the unbounded region whose closure contains only a_i and a_j. Let Δ be the open triangle determined by a_i $(i=1, 2, 3)$.

Case 1. If $x_3\in D(a_3)$, then $x_3a_3=A_1\cdot A_2$, since C_1 and C_2 lie in $\overline{\Delta}$, and the only convex subarc of A_1 and A_2 joining x_3 and a_3 must be a straight line segment. The line $L(x_3, a_3)$ must be a line of support to $H(C_1)$ and $H(C_2)$. Hence $L(x_3, a_3)\cdot R$ is a connected segment in $R\subset S$. Let $L(x)$ be a line of support to $H(C_3)$ at x. Then since $L(x_3, a_3)\cdot C_3\neq 0$, we have $L(x_3, a_3)\cdot L(x)\cdot R\equiv y$. Hence the convex polygonal arc $x_3y+yx\subset S$, and joins x to x_3. An almost identical argument holds if $x_3=a_3$.

Case 2. Suppose $x_3\in\overline{D}(a_1, a_3)$ with $x_3\neq a_3$. In this case $C_1=a_2a_3$, otherwise A_1 would not be convex since $C_1\subset\overline{\Delta}$. Recall that $x\in[A_3-A_3\cdot(A_1+A_2)]\subset\Delta$. Let $R^+(a_1, a_2)$ be the half-plane which contains a_3. Since $x_3\in\overline{D}(a_1, a_3)$ we have $x_3\in\overline{R}^+(a_1, a_2)$. Let $L(a_1, x)\cdot R\equiv\xi y$. Since $L(a_1, x)\cdot C_2=a_1$, we have $\xi y\subset R\subset S$. Moreover, $y\in a_2a_3$. The subarc of A_1 joining y to x_3 we call A_{1y}. The polygon $x_3x+xy\subset\overline{R}^+(x_3, a_1)$. Also $A_{1y}\subset\overline{R}^+(x_3, a_1)$. Hence it is easy to see that $xy+A_{1y}$ is a convex arc in S joining x to x_3.

Case 3. Suppose $x_3\in\overline{D}(a_1)$. As reasoned above, we must have here $C_1=a_2a_3$. On account of case 2 we may assume $x_3\notin L(a_1, a_2)$.

(α) If $x_3x\cdot(a_1a_2-a_1)=0$. Let $L(a_1, x)\cdot a_2a_3=y$. Then $xy\subset R\subset S$. Let $R^+(a_1, x)$ be the half-plane which contains a_3. Then $x_3\in\overline{R}^+(a_1, x)$. The polygonal line $x_3x+xy\subset\overline{R}^+(a_1, x)$. As before let A_{1y} be the subarc of A_1 joining y to x_3. Again $xy+A_{1y}$ is a convex arc lying in S joining x to x_3.

(β) If $x_3x\cdot(a_1a_2-a_1)\neq 0$, let $L(x_3, x)\cdot a_2a_3=y$. Since $L(x_3, x)\cdot L(x_3, a_1)=x_3$, we must have $L(x_3, x)\cdot C_2=0$. Hence $L(x_3, x)\cdot R=xy\subset R\subset S$. Let $R^+(x_3, a_2)$ be the half-plane which contains a_3. Then $x_3y\subset\overline{R}^+(x_3, a_2)$. Hence $xy+A_{1y}$ is a convex arc in S joining x_3 to x.

Case 4. If $x_3 \in \bar{D}(a_2, a_3)$ or if $x_3 \in \bar{D}(a_2)$, the proofs are exactly as above with appropriate notation. Moreover, $x_3 \notin D(a_1, a_2)$ and $x_3 \notin a_1a_2$, otherwise A_1 or A_2 would not be convex. Also $x_3 \notin \Delta$, otherwise A_1 or A_2 would not be a minimal convex arc. This completes the proof of Theorem 3.

If S is a simply connected continuum, then $K(S)$ is a simply connected arcwise convex continuum. To prove this, first observe that $K(S) \subset S$, so that $K(S)$ is bounded. The fact that $K(S)$ is closed follows easily from a theorem of Blaschke [4, p. 62] concerning the convergence of uniformly bounded sequences of convex sets. To prove that $K(S)$ is simply connected, suppose there exists a bounded component K_1 of the complement of $K(S)$. Since $K(S) \subset S$, and since S is simply connected, we have $\bar{K}_1 \subset S$. Choose $x \in K_1$. Since $K(S)$ is closed, let C be a line segment containing x and having only its end points a and b in $K(S)$. Since $\bar{K}_1 \subset S$, we have $ab \subset S$. Now since ab is the minimal convex arc in S joining a and b, and since $a \in K(S)$, $b \in K(S)$, the proof given for Theorem 3 implies that $ab \subset K(S)$. This together with Theorem 3 implies the above italicized statement.

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University of California at Los Angeles