ON THE IDEALS AND AUTOMORPHISMS OF NON-ASSOCIATIVE RINGS

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Introduction. Relations between the multiplication ring of a ring (=non-associative ring)[1] and the ring itself have been pointed out by a number of writers [1–7]. In the present note it is shown that the ideal lattice of a ring with unit element is isomorphic to the sublattice of all right ideals of the multiplication ring which contain the annihilator of the unit. A corresponding result is obtained for the right ideal lattice. These results generalize some conditions of R. D. Schafer [7] which describe the simplicity or right simplicity of an algebra with unit element in terms of the right ideal structure of the multiplication ring. We do not quite assume the existence of a unit. (See the paragraph preceding Lemma 2.) By adjoining a unit element, we are able to derive similar, but not quite so precise, results for arbitrary rings.

We also give considerably simplified proofs of the results[2] of Schafer [7] which concern the automorphisms of rings with unit. Here our simplification consists in avoiding the so-called "reconstruction" of a ring with unit from its multiplication ring. Again we do not quite require a unit.

1. **Ideals in rings.** Consider a ring A. Let M_r (M_l) denote the set of all right (left) multiplications R_a: $xR_a = xa$ $(L_a$: $xL_a = ax)$ of A. Let $M_r^* \equiv [\rho]$ $(M \equiv [\mu])$ denote the ring generated by M_r (by $M_r \cup M_l$). Let $N_r \equiv [\nu_r]$ $(N \equiv [\nu])$ be the right ideal of M_r^* (of M) which is generated by the set of elements of M_r^* of the form $\rho = R_{xy} - R_xR_y$ (of M of the forms $\mu = R_{xy} - R_xR_y$, $\mu = L_{xy} - L_yL_x$, and $\mu = R_x - L_x$), where x and y are arbitrary elements of A.

LEMMA 1. *If* $a \in A$, $\rho \in M_r^*$, *and* $\mu \in M$, *then* (1) $R_a\rho - R_{a\rho} \in N_r$ *and* (2) $R_a\mu - R_{a\mu} \in N$.

PROOF. The relations (1) and (2) are additive in ρ and in μ and we

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[1] Following a suggestion of an editor and a referee, we shall use hereafter in this note the terms *ring* and *algebra* in place of *non-associative ring* and *non-associative algebra*. The term *naring* of [8] will not be used here. Numbers in brackets refer to the references at the end of the paper.

[2] Schafer stated these results for algebras of finite order over a base field, but his arguments do not essentially use finite dimensionality except on p. 582, line 13, and in the proof of his Theorem 6.

may assume that ρ and μ are products of finitely many elements of M_r and of $M_r \cup M_l$, respectively. We then prove (1) and (2) by induction on the number of factors of ρ and of μ. If $\rho = R_x$, then (1) holds by the definition of N_r. To complete the induction, write $\rho = WR_y$, $R_a\rho - R_{a\rho} = R_a WR_y - R_{aWR_y} = R_{aW}R_y - R_{(aW)y} + (R_a W - R_{aW})R_y \in N_r$. If $\mu = L_x$, we obtain (2) by the following computation: $R_a L_x - R_{aL_x} = L_a L_x - L_{xa} + (R_a - L_a)L_x - (R_{aL_x} - L_{aL_x}) \in N$. To complete the induction, it suffices to note that if $\mu = WL_y$, then $R_a\mu - R_{a\mu} = R_a WL_y - R_{aWL_y} = R_{aW}L_y - L_{y(aW)} + (R_a W - R_{aW}) L_y = L_{aW}L_y - L_{y(aW)} + (R_{aW} - L_{aW})L_y + (R_a W - R_{aW})L_y \in N$.

Remark. We now see that if A has a unit element 1, then N_r (N) consists of all ρ (μ) such that $1\rho = 0$ ($1\mu = 0$). For clearly $1\rho = 0$ ($1\mu = 0$) if $\rho \in N_r$ ($\mu \in N$). Conversely, Lemma 1 with $a = 1$ shows that $\rho \in N_r$ ($\mu \in N$) if $1\rho = 0$ ($1\mu = 0$). Even if A has no unit element, it is easy to see that $c \in A$ is in the center of A if and only if $c\nu = 0$ for every $\nu \in N$ (cf. [7, Theorem 8]).

We shall say that A *satisfies condition* U_l (U) in case $R_x = 0$ implies that $x = 0$ and $M_r \cap N_r = 0$ ($M_r \cap N = 0$). We shall see that the condition U_l (U) will replace the assumption of the existence of a left unit element (unit element) in our discussion. That the condition U (U_l) is actually weaker than the outright assumption of a unit element (left unit element) may be seen by considering a Boolean ring which has no unit.

Remark. Nakayama [5] has also indicated a weakening of the requirement that A have a unit. He requires merely that every finite subset of A has a left unit in A and also a right unit in A. This condition implies our condition U_l, as is easily seen, but to obtain the condition U we seem to require that each finite subset of A has a *unit* in A.

Lemma 2. *If $\rho \in M_r^*$ ($\mu \in M$), then there are elements $a \in A$, $\nu_r \in N_r$ ($a \in A$, $\nu \in N$) such that $\rho = R_a + \nu_r$ ($\mu = R_a + \nu$). These elements are unique if A satisfies the conditon* U_l (U).

Proof. Again additivity permits us to assume that ρ (μ) is a product of finitely many elements of M_r (of $M_r \cup M_l$). But then $\rho = R_a$ or $\rho = R_a\rho_1$. In the latter case we have $\rho = R_{a\rho_1} + \nu_r$ by Lemma 1. To complete the proof, we must consider $\mu = L_a = R_a + (L_a - R_a) = R_a + \nu$, and $\mu = L_a\mu_1 = R_a\mu_1 + (L_a - R_a)\mu_1 = R_{a\mu_1} + \nu'$, by Lemma 1. The uniqueness under the stated conditions is obvious.

Theorem 1. *Let A be a ring which satisfies condition* U. *Then the mapping $I \to I^* \equiv [R_a + \nu;\ a \in I,\ \nu \in N]$ is a lattice isomorphism be-*

tween the ideal lattice of A and the lattice of all right ideals of M which contain N.

PROOF. Clearly I^* contains N. To see that I^* is a right ideal of M, note first that if $a, b \in I$, $\nu, \nu' \in N$, then $(R_a+\nu)-(R_b+\nu')=R_{a-b}+(\nu-\nu')\in I^*$. If $\mu \in M$, we use Lemmas 1 and 2 to compute $(R_a+\nu)\mu = (R_a+\nu)(R_x+\nu_1)=R_{ax}+R_aR_x-R_{ax}+R_{a\nu_1}+R_a\nu_1-R_{a\nu_1}+\nu R_x+\nu\nu_1 = R_{ax+a\nu_1}+\nu_2\in I^*$. Thus I^* is a right ideal of M. Now let $\mathfrak{J}$ be an arbitrary right ideal of M which contains N. Define $I\equiv[a;\ R_a\in\mathfrak{J}]$. Then we verify that I is an ideal of A as follows. Let $a, b\in I$, $x\in A$. Then $R_{a-b}=R_a-R_b\in\mathfrak{J}$, $R_{ax}=R_aR_x+R_{ax}-R_aR_x\in\mathfrak{J}$, since $\mathfrak{J}$ contains N, and $R_{xa}=L_{xa}+R_{xa}-L_{xa}=R_aL_x+(L_a-R_a)L_x+L_{xa}-L_aL_x+R_{xa}-L_{xa}\in\mathfrak{J}$, since $\mathfrak{J}$ contains N. Thus I is an ideal of A. It is easy to see that $I^*\leqq\mathfrak{J}$ and Lemma 2 assures us that $\mathfrak{J}\leqq I^*$. Thus $I^*=\mathfrak{J}$, and our mapping is exhaustive. To see that it is one-to-one, let $I^*=J^*$ for ideals I and J of A. If $a\in I$, then $R_a\in I^*=J^*$, $R_a=R_b+\nu$ with $b\in J$ and $\nu\in N$. Then $R_{a-b}\in N\cap M_r$, and the condition U yields $a=b\in J$. We have shown that $I\leqq J$, and an interchange of I and J yields $I=J$. To complete the proof, it suffices to remark that if $I\geqq J$, then $I^*\geqq J^*$ and that $I=[a;\ R_a\in I^*]$ shows that $I^*\geqq J^*$ implies that $I\geqq J$. Thus our mapping $I\rightarrow I^*$ is one-to-one and preserves order, and is, therefore, a lattice isomorphism.

REMARK. The anti-isomorph of a simple example previously given by us [8] shows that Theorem 1 need not be true when the condition U is suppressed. This example is an algebra A (over a field F) with basal units e and u and multiplication table: $ee=uu=e$, $eu=0$, $ue=u$. It was shown in [8] that A is simple. With respect to the given basis, the right multiplications of A are represented by matrices of the form

$$R_x = \begin{bmatrix}\alpha & 0\\ \beta & \alpha\end{bmatrix} \qquad (x=\alpha e+\beta u;\ \alpha, \beta\in F).$$

Thus $M_r=M_r^*$ in our example. If $x'=\alpha' e+\beta' u$, with $\alpha', \beta'\in F$, then $R_{xx'}-R_xR_{x'}=\beta' R_y$, where $y=\beta e-\alpha u$. Thus $N_r=M_r=M_r^*$. Since we always have $N\geqq N_r$, it is clear that A satisfies neither condition U nor condition U_l. Since

$$L_x = \begin{bmatrix}\alpha & \beta\\ \beta & 0\end{bmatrix},$$

we easily see that $M\cong F_2$, a total matric ring over F. Then we see that $N=M$, by the known form of right ideals of F_2. The conclusion of Theorem 1 does not hold for A. Our example shows also that our next theorem is false when the condition U_l is suppressed.

THEOREM 2. *Let A be a ring which satisfies the condition* U_l. *Then the mapping* $I \rightarrow I^+ \equiv [R_a + \nu_r;\ a \in I,\ \nu_r \in N_r]$ *is a lattice isomorphism between the right ideal lattice of A and the lattice of all right ideals of M_r^* which contains N_r.*

We shall not give the proof, since it is almost a verbatim repetition of the proof of Theorem 1.

It is well known that we may embed an arbitrary ring A into a ring A' which has a unit element and is such that A is an ideal of A'. This may be done, of course, in infinitely many different ways. It should be observed that the usual construction also yields the fact that every (right) ideal of A is a (right) ideal of A'. Then we may obtain a lattice isomorphism of the (right) ideal lattice of A and the sublattice of all right ideals of M' (of $(M_r')^*$) which contain N' (N_r') and which are contained in the right ideal $[R_a + \nu';\ a \in A,\ \nu' \in N']$ of M' (the right ideal $[R_a + \nu_r';\ a \in A,\ \nu_r' \in N_r']$ of $(M_r')^*$).

2. **Automorphisms of rings.** Let A be a ring which satisfies condition U_l and let S^* be an automorphism of M_r^* such that $N_r S^* = N_r$ and $M_r S^* = M_r$. Then, if $x \in A$, we may define xS uniquely by means of the equation $R_x S^* = R_{xS}$. We shall prove that the mapping S: $x \rightarrow xS$ is an automorphism of A. Clearly $AS = A$, since $M_r S^* = M_r$. Also $xS = yS$ gives $R_x S^* = R_y S^*$, $R_x = R_y$, $x = y$ by condition U_l. Thus S is a one-to-one mapping of A onto A. To see that S is an automorphism of A, first compute $R_{xS+yS} = R_{xS} + R_{yS} = R_x S^* + R_y S^* = (R_x + R_y)S^* = (R_{x+y})S^* = R_{(x+y)S}$. Then condition U_l gives $(x+y)S = xS + yS$. Observe that $(R_{xy} - R_x R_y)S^* = R_{xy}S^* - (R_x S^*)(R_y S^*) = R_{(xy)S} - R_{xS}R_{yS} \in N_r$, since $N_r S^* = N_r$. Now we see that $R_{(xy)S} - R_{(xS)(yS)} \in N_r$. Finally, condition U_l gives $(xy)S = (xS)(yS)$, and S is an automorphism of A.

If S_1^* is an automorphism of M which maps each of the sets N, M_r, and M_l onto itself, then $N_r S_1^* = N_r$, provided that the condition U is valid. To see this, note that $(R_{xy} - R_x R_y)S_1^* = R_{xy}S_1^* - (R_x S_1^*) \cdot (R_y S_1^*) = R_u - R_v R_w = R_u - R_{vw} + \nu_r$, where u, v, $w \in A$ and $\nu_r \in N_r$. Condition U yields $u - vw = 0$, $(R_{xy} - R_x R_y)S_1^* \in N_r$, and it follows that $N_r S^* \leqq N_r$. Since $(S_1^*)^{-1}$ is also an automorphism of M satisfying our requirements, we find also that $N_r(S_1^*)^{-1} \leqq N_r$, so that $N_r S_1^* = N_r$, as desired.

THEOREM 3. *Let G be the group of automorphisms S of a ring A which satisfies condition* U. *Let H be the group of automorphisms of M which map each of the sets N, M_r, and M_l onto itself. Then the mapping*

$S^*\colon \mu\to\mu S^* = S^{-1}\mu S$ is in H for every $S\in G$, and the mapping $S\to S^*$ is an isomorphism of G onto H.

Proof. It is clear that S^* is an automorphism of M and that $(S^*)^{-1}=(S^{-1})^*$. To prove that S^* maps M_r onto itself, note that $R_aS^*=S^{-1}R_aS=R_{aS}\in M_r$, so that $M_rS^*\leqq M_r$, and $M_r(S^*)^{-1}\leqq M_r$, $M_rS^*=M_r$. The proof that $M_lS^*=M_l$ is dual. The generators of N are carried by S^* into N, so that $NS^*\leqq N$, $N(S^*)^{-1}\leqq N$, $NS^*=N$. We have proved that $S^*\in H$.

Now let $\Sigma\in H$. Then our remarks preceding the statement of Theorem 3 show that $R_x\Sigma=R_{xS}$ defines an automorphism $x\to xS$ of A. Then we have $R_{xS}=S^{-1}R_xS=R_xS^*=R_x\Sigma$ for every $x\in A$. Dually, we find that $L_x\Sigma=L_{xT}$ defines an automorphism $x\to xT$ of A, and $L_{xT}=T^{-1}L_xT=L_xT^*=L_x\Sigma$. Then $S=T$, since $R_x-L_x \in N$, $(R_x-L_x)\Sigma\in N$, $(R_{xS}-L_{xT})\in N$, $R_{xS}-R_{xT}\in N$, $xS=xT$ for every $x\in A$ by condition U. Hence $L_x\Sigma=L_xS^*$, and $\mu\Sigma=\mu S^*$ for every $\mu\in M$ follows readily since Σ is an automorphism of M. We have proved that the mapping $S\to S^*$ is onto H. That this mapping is a homomorphism is a trivial consequence of its definition. The kernel consists of those elements $S\in G$ such that $S^*=I$, $R_x=R_xS^*=S^{-1}R_xS =R_{xS}$, $x=xS$ for every $x\in A$ by condition U, $S=I$. Thus the maping $S\to S^*$ is actually an isomorphism and the proof is complete.

The following theorem is proved in similar fashion.

Theorem 4. *Let G be the group of automorphisms S of a ring A which satisfies condition U_l. Let H be the group of automorphisms of M_r^* which map each of the sets N_r and M_r onto itself. Then the mapping S^*: $\rho\to\rho S^*=S^{-1}\rho S$ is in H for every $S\in G$, and the mapping $S\to S^*$ is an isomorphism of G onto H.*

We conclude with the following theorem on inner automorphisms of M.

Theorem 5. *Let A be a ring which satisfies condition* U *and let S be an automorphism of A. Then there exists a nonsingular $\mu_0\in M$ such that $S^{-1}\mu S=\mu_0^{-1}\mu\mu_0$ for every $\mu\in M$ (if and) only if $S\in M$.*

Proof. Let the requirement of the theorem hold. Set $\mu=R_x$, and use $S^{-1}R_xS=R_{xS}$, $\mu_0^{-1}=R_a+\nu$ (valid by Lemma 2), to obtain $R_{xS} =(R_a+\nu)R_x\mu_0=R_aR_x\mu_0+\nu_1+R_y+\nu_2$, where $y=aR_x\mu_0$, and we have used Lemma 1. Now apply condition U to find that $xS=aR_x\mu_0 =xL_a\mu_0$, $S=L_a\mu_0\in M$, as desired.

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