A FACTORING THEOREM FOR HOMOMORPHISMS

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1. **The general factoring theorem.** The purpose of this note is to prove a general factoring theorem (Theorem 2) for homomorphsms of general algebras, and to show its usefulness by proving the Chevalley-Jacobson density theorem for irreducible rings of endomorphisms (Theorem 3). Theorem 2 is probably true for a very general class of algebraic structures, in the sense of Bourbaki [2, §8],[1] Birkhoff [1], Jónsson-Tarski [6], or as outlined in [3]. However, we prove it here only for a class of algebras A which have a binary operator $+$ and a neutral element 0, under which A is a group, and which share a class Ω of operators α, applicable to finite or transfinite sequences of elements of A, and such that $\alpha(0, 0, \cdots)=0$. These restrictions imply that $\{0\}$ is a subalgebra of A, and allow the definition of "inner" direct sums of subalgebras. In what follows, "algebra" will always refer to such an algebraic structure.

We begin by stating the following well known elementary factoring theorem.

THEOREM 1. *Let A, B, and C be algebras. Let ϕ and ψ be homomorphisms of A onto B, and A into C, respectively. Let* kern (ϕ) $\subset$kern (ψ). *Then, there is a (unique) homomorphism θ of B into C such that $\psi=\theta\phi$.*

One need only choose θ as $\{(\phi(x), \psi(x)) \mid x\in A\}$, a subset of $B\times C$. If $\phi(x)=\phi(x')$, then $x-x'\in$kern (ϕ) so that $x-x'\in$kern (ψ), and $\psi(x)=\psi(x')$; thus, θ is a function mapping B into C and such that $\psi(x)=\theta(\phi(x))$ for all $x\in A$. The verification that θ is a homomorphism is easily made.

Our principal theorem is a generalization of this, with one additional restriction. We recall that an algebra B is said to be simple (or irreducible) if B contains no proper subalgebras.

THEOREM 2. *Let A, B_1, B_2, $\cdots$, B_n, and C be algebras, with the B_i simple. Let ϕ_i be a homomorphism of A onto B_i and let ψ be a homomorphism of A into C; let $K_i=$*kern (ϕ_i), *and $K=$*kern (ψ). *Then, the condition $\bigcap K_i\subset K$ is necessary and sufficient for the existence of homomorphisms θ_i of B_i into C such that $\psi=\sum\theta_i\phi_i$.*

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[1] Numbers in brackets refer to the references at the end of the paper.

We shall say that a finite collection $\{S_1, S_2, \cdots, S_r\}$ of sets is "free" if no S_i contains the intersection of the remaining sets. Given any finite collection $\{S_1, S_2, \cdots, S_n\}$, it is possible to relabel these in such a manner that for some $r \geqq 1$, the subset $\{S_1, S_2, \cdots, S_r\}$ is free, while if $i>r$, $S_1 \cap S_2 \cap \cdots \cap S_r \subset S_i$. Doing this to the collection $\{K_1, \cdots, K_n\}$, we may assume that $\{K_1, \cdots, K_r\}$ is free, and that $K_1 \cap \cdots \cap K_r \subset K$. Let $B = PB_i$, the "outer" direct product of the algebras $B_1, \cdots, B_r$. B is again such an algebra, and may be regarded as the "inner" direct sum of subalgebras $\overline{B}_i$, isomorphic to the factors B_i. Define a function Φ by: $\Phi(x) = (\phi_1(x), \phi_2(x), \cdots, \phi_r(x))$; this is a homomorphism of A into B, and its kernel is exactly $\bigcap_1^r K_i$. Thus, we have two homomorphisms, Φ and ψ, of A into B and into C, with kern $(\Phi) \subset$ kern (ψ). In order to apply Theorem 1, we must show that Φ is onto. Let $\overline{K}_j = \bigcap'_{i \leqq r} K_i$, the prime indicating that K_j is omitted; this is a subalgebra of A, which is mapped by Φ into the jth summand $\overline{B}_j$. Since $\overline{K}_j$ is not contained in K_j, $\Phi(\overline{K}_j)$ is nonzero, and since B_j is simple, must be $\overline{B}_j$ itself. Thus, $\Phi(A)$ contains each of the summands $\overline{B}_j$, and must therefore be B. By Theorem 1, we conclude that there is a homomorphism θ of B into C such that $\psi = \theta\Phi$. Splitting θ into its components θ_i, acting on the factors B_i of B, we have $\psi = \theta\Phi = \sum \theta_i \phi_i$, where θ_i is a homomorphism of B_i into C. This proves the sufficiency; the necessity is obvious.

We observe that a corresponding theorem may be proved in which the requirement that the algebras B_i be simple is relaxed to the condition that each is a finite direct sum of simple algebras. The proof is the same, since $B = PB_i$ is again a direct sum of simple algebras. The same can be said if B is such that any homomorphism of a subalgebra of B into C may be extended to a homomorphism of B itself into C.

2. **The Chevalley-Jacobson density theorem.**

As an application of Theorem 2, we shall obtain a proof of the fundamental density theorem for irreducible rings of endomorphisms [**4**, **5**, **7**].

THEOREM 3. *Let R be an irreducible subring of the ring $E(V)$ of all endomorphisms of the abelian group V. Let $D = L_R(V)$, the ring of all R-linear endomorphisms of V. Then R is dense in $L_D(V)$.*

What has to be shown is that if $a_1, \cdots, a_{n+1}$ is any finite ordered set of D-independent points of V, then there is an element x of R such that $x(a_1) = \cdots = x(a_n) = 0$, while $x(a_{n+1}) \neq 0$. We suppose that no such x exists. Define functions ϕ_i and ψ by: $\phi_i(x) = x(a_i)$, $\psi(x) = x(a_{n+1})$. Then, ϕ_i and ψ are homomorphisms of R^+, the additive group of R, onto V, with the ring R as a ring of operators. Since R is irreducible, V is simple. By the assumption above, $\phi_i(x)=0$ for $i\leqq n$ implies that $\psi(x)=0$, so that $\bigcap K_i\subset K$. If Theorem 2 is applied, there exist homomorphisms θ_i of V into V—whence $\theta_i\in D$—such that $\psi=\sum\theta_i\phi_i$. This in turn implies that $a_{n+1}=\sum\theta_i a_i$, which contradicts the D-independence of $a_1, \cdots, a_{n+1}$.

References

1. G. Birkhoff, *On the structure of abstract algebras*, Proc. Cambridge Philos. Soc. vol. 31 (1935) pp. 433–454.
2. N. Bourbaki, *Éléments de mathématique* (*Theorie des ensembles*), Actualités Scientifiques et Industrielles, no. 846, Paris, 1939.
3. R. C. Buck, *Canonical mappings and algebraic structures*, Bull. Amer. Math. Soc. Abstracts 55-7-331, 55-7-332.
4. N. Jacobson, *Structure of simple rings without finiteness assumptions*, Trans. Amer. Math. Soc. vol. 57 (1945) pp. 228–245.
5. ———, *The radical and semi-simplicity for arbitrary rings*, Amer. J. Math. vol. 67 (1945) pp. 300–320.
6. B. Jónsson and A. Tarski, *Direct decompositions of finite algebraic systems*, Notre Dame Mathematical Lectures. no. 5, 1947.
7. T. Nakayama, *Über einfache distributive Systeme unendlicher Ränge*, Proc. Imp. Acad. Tokyo vol. 20 (1944) pp. 61–66.

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