AN EXTENSION OF THE JACOBSON RADICAL

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A *cluster* [1][1] is an additively written group closed under a multiplication which is right and left distributive with respect to the addition. A *naring* is a cluster whose additive group is abelian. A ring, then, is a naring whose multiplication is associative. The purpose of this note is to extend to an arbitrary cluster certain parts of N. Jacobson's theory of the radical [2] of a ring, even though current interest in such an extension may seem to center in the naring case.

Let R be a cluster and let R_+ denote its additive group. Among the endomorphisms of R_+ are the right multiplications $x \to xa$ and the left multiplications $x \to ax$. Let Ω_r denote the set of endomorphisms of R_+ consisting of all inner automorphisms and all right multiplications. Let Ω consist of Ω_r and all left multiplications. Then R_+ is simultaneously an Ω-group (see, for example, [3, pp. 4–6]) and an Ω_r-group. The ideals (right ideals) in R, as defined in [1], are just the Ω-subgroups (Ω_r-subgroups) of R_+.

The ideal in R generated by an element a of R is denoted by (a). The sum of two ideals is the least ideal containing them both. It is an immediate consequence that if $a, b \in R$, then $(a+b) \subseteq (a)+(b)$.

If I is a right ideal in R, it follows readily that the set $I' = \{a \in R; (a) \subseteq I\}$ is an ideal in R, in fact the greatest ideal in R contained in I. It is this property of the ideal I' which permits its use in a cluster as a substitute for the quotient ideal $I:R = \{a \in R; Ra \subseteq I\}$ of the ring theory. For *if I is a right ideal in a ring R and $I:R \subseteq I$, then $I' = I:R$.*

If a is an element of a cluster R, let $Q(a)$ denote the right ideal in R generated by the set $\{ax - x; x \in R\}$. Let $(b)_r$ denote the right ideal in R generated by the element b. The following lemma is implied by the fact[2] that

$$(a + b)x - x = (ax - x) + (x + bx - x) \in Q(a) + (b)_r.$$

LEMMA 1. *If $a, b \in R$, then $Q(a+b) \subseteq Q(a) + (b)_r$.*

An element a of R is *quasi-regular* if and only if $a \in Q(a)$, while a subset of R is quasi-regular if and only if each of its element is. Evidently quasi-regularity of elements and of ideals is preserved under cluster homomorphism.

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] For this and other helpful observations, we are indebted to the referee.

The *radical* N of a cluster R is defined to be the set $N=\{a\in R;\ (a)$ is quasi-regular$\}$. Under cluster homomorphism, N maps into the radical of the homomorph.

A right ideal I in a cluster R is *modular*[3] if and only if R contains an element e such that $ex-x\in I$ if $x\in R$. In this case, if $e\in I$, then $I=R$. If I is a modular right ideal in a ring R, then $I:R\subseteq I$, and hence $I'=I:R$.

THEOREM 1. *The radical N of a cluster R is the intersection X of all ideals M' in R such that M is a modular maximal right ideal in R.*

PROOF. We show first that $X\subseteq N$. If a is not in N, then some b in (a) is not in $Q(b)$. By Zorn's lemma, there is a right ideal M maximal in the set of all right ideals in R which contain $Q(b)$ but not b. Now M is modular since $bx-x\in Q(b)\subseteq M$, $x\in R$; and M is maximal right in R since any right ideal in R which properly contains M, contains b also, and so contains all of R. Moreover $M'\subseteq M$, so b is not in M'. Thus a is not in M', so a is not in X. Hence $X\subseteq N$.

Conversely, $N\subseteq X$. For if a is not in X, there is a modular maximal right ideal M such that a is not in M'. Thus some b in (a) is not in M, so $R=(b)_r+M$. Since M is modular, there is an element e of R such that $ex-x\in M$ for x in R, and e is not in M. Hence $e=c+m$ for some c in $(b)_r$ and m in M. Suppose that $a\in N$. Since $c\in(a)$, we have $c\in Q(c)$. Hence by Lemma 1, $e-m\in Q(e-m)\subseteq Q(e)+(m)_r\subseteq M$. Thus $e\in M$, a contradiction. It follows that a is not in N and that $N\subseteq X$, completing the proof.

The theorem shows that N is an ideal in R, and it follows easily that N is the greatest quasi-regular ideal in R.[4] We turn now to the residue-class cluster R/N.

THEOREM 2. *The cluster R/N has zero radical.*

PROOF. Denoting natural homomorphic images in R/N by bars, suppose $\bar{a}$ to be in the radical of $\overline{R}=R/N$, and that $b\in(a)$. Then $\bar{b}\in(\bar{a})$, so $\bar{b}\in Q(\bar{b})=\overline{Q(b)}$. Hence $b-c\in N$ for some c in $Q(b)$. Using Lemma 1, $b-c\in Q(b-c)\subseteq Q(b)+(c)_r\subseteq Q(b)$. Thus $b\in Q(b)$, $a\in N$, $\bar{a}=\bar{0}$, as required.

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[3] Called *regular* by I. E. Segal in a group algebra. The term *modular* is used in some unpublished notes of Jacobson on rings.

[4] This fact and Theorem 2 were obtained for a naring (without the use of Zorn's lemma) with N. H. McCoy in 1948, as well as the example following Theorem 3. The revival of interest which led to Theorem 1 is due to a recent letter from M. F. Smiley to McCoy proving an intersection theorem for N in a naring somewhat restricted in the direction of associativity.

It follows from this theorem that N is the least ideal B in R such that R/B has zero radical.

A cluster R is *primitive* if and only if R contains a modular maximal right ideal M such that $M'=0$. It is known that if a ring R contains a maximal right ideal M such that $M:R=0$, it contains also a modular maximal right ideal with the same property. Thus the notion of primitive cluster generalizes that of primitive ring [2].

It follows from Theorem 1 that a primitive cluster has zero radical, and it may be verified that if M is a modular maximal right ideal in a cluster R, then R/M' is primitive. It is clear that a direct sum of clusters is a cluster, and that the basic theorem on subdirect sums holds for clusters as it does for rings (see, for example, [4, p. 48]) and narings [6]. This is used to prove the following theorem.

THEOREM 3. *A cluster $R\neq 0$ is isomorphic to a subdirect sum of primitive clusters if and only if $N=0$.*

PROOF. Suppose that R is isomorphic to a subdirect sum of the primitive clusters R_k. Then R contains a set of ideals B_k such that $R/B_k\cong R_k$ and $\bigcap B_k=0$. Under the natural homomorphism of R onto R_k, N maps into the radical of R_k, which is zero since R_k is primitive. Hence $N\subseteq B_k$, and so $N\subseteq\bigcap B_k=0$. Conversely, if $N=0$, Theorem 1 shows that R is isomorphic to a subdirect sum of the primitive clusters R/M', where M ranges over the modular maximal right ideals in R, and the proof is finished.

The preceding theory has been obtained by the use of right ideals, and the terms quasi-regular, radical, primitive have been used for convenience in place of the more precise right quasi-regular, right radical, right primitive.

By using left ideals instead of right ideals, one obtains a left radical N_1 and the concept of a left primitive cluster. An example, used by Smiley [6] in a different connection, is given of a naring R in which $N_1\neq N$. Let R be the nonassociative algebra over, say, the rational field, with basis elements e, u such that $ee=e$, $eu=u$, $ue=0$, $uu=e$. Then R is simple, and since e is a left unit element, $N=0$. Now let $\alpha e+\beta u$ be any fixed element of R, and let I denote the left ideal in R generated by the set $\{x(\alpha e+\beta u)-x;\ x\in R\}$. Clearly I contains $u(\alpha e+\beta u)-u=\beta e-u$, and hence also contains $-\gamma u(\beta e-u)=\gamma e$ for any rational γ. Furthermore $e(\alpha e+\beta u)-e=(\alpha-1)e+\beta u$ is in I, so $\beta u\in I$. It follows that $\alpha e+\beta u\in I$, and hence $N_1=R$. It may now be noted also that R, being subdirectly irreducible, is a primitive naring which is not left primitive.

A generalization of the present theory, which contains as another special case the extension to a cluster of the theory of the F-radical of a naring [6], is under preparation jointly with McCoy [5].

Bibliography

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