

VECTOR SUBSERIES OF MAXIMUM MODULUS

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This note extends to Euclidean n -space E_n for $n \geq 2$ and to the generalized Euclidean space E_∞ certain investigations undertaken in [1]¹ with reference to E_2 . Among other things, we shall show that, given any absolutely convergent series $\sum c_k$ of elements in a Banach space, there is a subseries $\sum c_j^*$ of maximum modulus; that in E_n for $n \geq 2$ this maximum is greater than $\rho \sum |c_k|$, where the "best possible" value for ρ is $\Gamma(n/2) / \{2\pi^{1/2}\Gamma[(n+1)/2]\}$; and that in E_∞ the corresponding quantity ρ is zero. We shall also obtain certain related results in E_n and in E_∞ for series $\sum b_k$ with $\sum |b_k| = \infty$.

The value of ρ in E_n is not new, although its inaccessibility seems to have been ignored. It was obtained in 1938 by A. E. Mayer [2] for finite series, and again in 1943 by H. Hadwiger [3] as a corollary to a general theorem on "direction-functionals."

Since many of our present proofs are strictly parallel to proofs in [1], we shall make our symbols conform as closely as possible to those in that paper, shall use freely the theory of multi-dimensional analytic geometry, and shall generally omit proofs where the corresponding proofs in [1] are sufficiently suggestive. (Many of these proofs can be constructed from those in [1] by merely replacing the variable angle ϕ by a variable unit vector and the element of integration $d\phi$ by the element of n -dimensional solid angle.) We shall denote components of vectors by superscripts on the symbols for the vectors themselves, let x represent a variable unit vector, use the summation convention with respect to repeated *superscripts* (thus $x^i b_k^i$ signifies $\sum_i x^i b_k^i$), use abbreviations such as $B_k \equiv |b_k|$ and $(x, b_k) \equiv \arccos(x^i b_k^i / B_k)$, and insert a "(+)" before a summand or integrand to denote summation or integration over positive values only. Unless we specifically assert that any particular discussion relates to E_n or to E_∞ , it is to be assumed to relate to both.

Let \mathfrak{A} be the class of absolutely convergent series, finite or infinite, $\sum a_k$ (denoted alternatively by A) the class of nonzero vectors in E_n ($n \geq 2$) or in E_∞ , and $\sum a_j^i$ (denoted alternatively by S) the general subseries of $\sum a_k$ for fixed $\sum a_k$. Let \mathfrak{B} be the class of series $\sum b_k$ (denoted alternatively by B) of nonzero vectors, in the same space, for which $\sum |b_k| = \infty$, and $\sum b_j^i$ (denoted alternatively by T) the general subseries of $\sum b_k$ for fixed $\sum b_k$.

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¹ Numbers in brackets refer to the references cited at the end of the paper.

THEOREM 1. *Given an absolutely convergent series $\sum c_k$ of elements in a Banach space, there exists a subseries $\sum c_j^*$ of maximum modulus.*

PROOF. If the proposition is false, then there exists a sequence of subseries $\sum_j c_j^{(\nu)}$ ($\nu = 1, 2, \dots$) whose moduli tend to the (finite) supremum of the moduli of all subseries $\sum c_j^*$ of $\sum c_k$. Let successive subsequences of this sequence be chosen so that: in the first, c_1 is a term in each series or in none; in the second, c_2 is a term in each series or in none; and so on. Defining ϵ_n as 1 if c_n appears in the series of each subsequence after the n th step in this process, and as 0 otherwise, we conclude that the modulus of each series in the n th subsequence differs from that of $\sum \epsilon_k c_k$ by not more than $\sum_{k>n} |c_k|$; and since this tends to zero as n becomes infinite, we see that $|\sum \epsilon_k c_k|$ equals the aforementioned supremum. The conclusion follows.

Returning to E_n and E_∞ , we define

$$F(x) \equiv \sum (+)A_k \cos(x, a_k) \quad \text{for all } x.$$

In what follows, up to and including Theorem 3, $\sum a_k$ will signify an arbitrary but fixed series of class \mathfrak{A} .

THEOREM 2. *Let $\sum \bar{a}_j$ be a subseries of $\sum a_k$ for which $|\sum \bar{a}_j| = \max_S |\sum a_j^*|$, and let $\bar{x} \equiv \sum \bar{a}_j / |\sum \bar{a}_j|$. Then $\sup F(x) = |\sum \bar{a}_j| = F(\bar{x})$.*

PROOF.² Let x be arbitrary and $\sum a_j^*$ the sum of those terms of $\sum a_k$ for which $\cos(x, a_k) > 0$. Then

$$\begin{aligned} F(\bar{x}) &= \sum (+)A_k \cos(\bar{x}, a_k) \geq \sum \bar{A}_j \cos(\bar{x}, \bar{a}_j) = |\sum \bar{a}_j| \geq |\sum a_j^*| \\ &\geq \sum (+)A_k \cos(x, a_k) = F(x). \end{aligned}$$

COROLLARY 2.1. *In the notation of Theorem 2, $\sum \bar{a}_j$ is the sum of those terms of $\sum a_k$ for which $\cos(\bar{x}, a_k) > 0$.*

THEOREM³ 2a. *Let x^* be such that $F(x^*) = \max F(x)$, and let $\sum a_j^*$ be the sum of those terms of $\sum a_k$ for which $\cos(x^*, a_k) > 0$. Then $\max_S |\sum a_j^*| = F(x^*) = |\sum a_j^*|$.*

² We insert this proof only to demonstrate the sense in which henceforth omitted proofs are essentially identical with the corresponding proofs in [1]. Those who are unfamiliar with the techniques of multi-dimensional analytic geometry will observe that the "geometrically obvious" inequality $|\sum a_j^*| \geq \sum (+)A_k \cos(x, a_k)$ is merely the Cauchy-Schwartz inequality $[(\sum a_j^*) / |\sum a_j^*|]^2 (\sum a_j^*)^2 \geq x^2 (\sum a_j^*)^2$.

³ This is the counterpart of Theorem 1 of [1]. Its inclusion after the present Theorem 1 is due to the fact that we cannot show by the method of [1] that $F(x)$ attains its supremum in E_∞ , since E_∞ is not locally compact.

COROLLARY 2a.1. *In the notation of Theorem 2a, $x^* = \sum a_j^* / |\sum a_j^*|$.*

THEOREM 3. *In the notation of Theorem 2, there exists no term a_i of $\sum a_k$ for which $\cos(x, a_i) = 0$.*

We now define $\rho \equiv \inf_A \max_S |\sum a_j^*| / \sum A_k$.

THEOREM 4. *In $E_n, \rho = \Gamma(n/2) / \{2\pi^{1/2}\Gamma[(n+1)/2]\}$; in $E_\infty, \rho = 0$.*

PROOF. In E_∞ we let the positive integer N be arbitrary and consider the series of N terms, $\sum a_k \equiv (1, 0, 0, \dots, 0, 0, 0, \dots) + (0, 1, 0, \dots, 0, 0, 0, \dots) + (0, 0, 1, \dots, 0, 0, 0, \dots) + \dots + (0, 0, 0, \dots, 0, 1, 0, \dots)$. It is clear that the subseries $\sum a_j^*$ of maximum modulus is $\sum a_k$ itself; and we have $|\sum a_j^*| / \sum A_k = N^{1/2} / N = 1 / N^{1/2}$.

In E_n we integrate $F(x)$ with respect to ω over S , where ω is the solid angle, whose element $d\omega \equiv dx^2 dx^3 \dots dx^n / x^1$ ($\sum x^{i^2} = 1$) is invariant under rotation, and where S is the surface of the unit sphere. Thus $\int_S F(x) d\omega = \sum A_k \int_S (+) \cos(x, a_k) d\omega$. But if for each k we introduce a rotation $\xi^i = b_j^i x^j$ in which $b_1^i = a_k^i / A_k$, we have $\int_S (+) \cos(x, a_k) d\omega = \int_S (+) \xi^1 d\omega = \int_{S: \xi^1 > 0} d\xi^2 d\xi^3 \dots d\xi^n = V_{n-1}$, where V_{n-1} is the volume of the unit sphere in $(n-1)$ -space. So $\int_S F(x) d\omega = V_{n-1} \sum A_k$. And since also $S = \int_S d\omega$, we have $\max F(x) \geq \sum A_k (V_{n-1} / S)$, so that, by Theorem 2a, $\rho \geq V_{n-1} / S$, which has the value asserted for ρ in the theorem [4].

We now divide S into portions $\Delta_k S$ of maximum diameter δ , choose for each k the position vector ξ_k of an arbitrary point in $\Delta_k S$, and consider the finite series $\sum a_k \equiv \sum \xi_k \Delta_k S$. Then $F(x) = \sum (+) \cos(x, \xi_k) \Delta_k S$. Letting $\epsilon > 0$ be arbitrary, we choose δ such that

$$(1) \quad \left| \sum (+) \cos(x, \xi_k) \Delta_k S - \int_S (+) \cos(x, \xi) d\omega \right| < \epsilon$$

for all x . [To see that this is possible we observe that for each x a $\delta \equiv \delta(x)$ exists for which (1) is true and then deduce the desired uniformity by contradiction.] But under a rotation from ξ^i coordinates to η^j coordinates in which $\eta^1 = x^i \xi^i$ the integral in (1) reduces to V_{n-1} , and since $\sum A_k = \sum |\xi_k \Delta_k S| = S$, we have $|F(x) - \sum A_k (V_{n-1} / S)| < \epsilon$ for all x . This completes the proof.

THEOREM 5. *There exists no series $\sum a_k \in \mathfrak{A}$ for which $F(x)$ is constant.*

THEOREM 6. *Given an arbitrary sequence, finite or infinite, of pairs (C_k, x_k) , where the x_k are unit vectors and the C_k are positive numbers with $\sum C_k < \infty$, then $\Phi(x) \equiv \sum C_k |x^i x_k^i|$ is not constant.*

PROOF. The series $\sum a_k$ defined thus:

$$a_{2k-1} \equiv C_k x_k, a_{2k} \equiv -C_k x_k$$

is of class \mathfrak{A} , and

$$\begin{aligned} F(x) &= \sum (+)C_k \cos(x, x_k) + \sum (+)[-C_k \cos(x, x_k)] \\ &= \sum C_k |x^i x_k^i| = \Phi(x). \end{aligned}$$

The conclusion now follows from Theorem 5.

THEOREM 7. *There exists no series $\sum a_k \in \mathfrak{A}$ for which it is true that $\max_S | \sum a_j' | / \sum A_k = \rho$.*

PROOF. In E_∞ this is obvious. In E_n the proof proceeds as in [1].

THEOREM 8. *Let $\sum b_k \in \mathfrak{B}$ be arbitrary. There exists a subseries $\sum b_j^*$ of $\sum b_k$ for which⁴*

$$\begin{aligned} \limsup_N | \sum' b_j^* | / \sum_1^N B_k &= \sup_T \limsup_N | \sum' b_j' | / \sum_1^N B_k \\ &= \limsup_N \sup_T | \sum' b_j' | / \sum_1^N B_k \\ &= \limsup_N \max_T | \sum' b_j' | / \sum_1^N B_k. \end{aligned}$$

We now define $\sigma \equiv \inf_B \max_T \limsup_N | \sum' b_j' | / \sum_1^N B_k$.

LEMMA 9.1. $\sigma \geq \rho$.

Before proving Lemma 9.2 (below), we make the following constructions.

In E_∞ we define $b_1 \equiv (1, 0, 0, \dots)$, $b_2 \equiv (0, 1, 0, \dots)$, \dots .

In E_n , for each $\nu = 1, 2, \dots$, we divide S into portions $\Delta_k^{(\nu)} S$ of maximum diameter $\delta^{(\nu)}$, choose for each k the position vector $\xi_k^{(\nu)}$ of an arbitrary point in $\Delta_k^{(\nu)} S$, and let $\delta^{(\nu)}$ be so small that

$$(2) \quad \left| \sum (+) x^i \xi_k^{(\nu) i} \Delta_k^{(\nu)} S - \int_S (+) x^i \xi^i d\omega \right| < S/2^\nu \quad \text{for all } x.$$

(Compare the proof of Theorem 4.) Let N_ν be the number of portions into which S is thus divided and define the series $\sum b_k$ of class \mathfrak{B} in such a way that the first N_1 terms are the $\xi_k^{(1)} \Delta_k^{(1)} S$ in any order, the next N_2 terms are the $\xi_k^{(2)} \Delta_k^{(2)} S$ in any order, and so on.

⁴ The notation \sum' indicates summation over precisely those terms of the subseries which occur among the terms of the original series summed elsewhere in the formula.

We now define, for each x ,

$$F_N(x) \equiv \sum_1^N (+) b_k^i x^i / \sum_1^N \bar{B}_k.$$

LEMMA 9.2. $\lim F_N(x) = \rho$ uniformly.

PROOF. In E_∞ we let N be any positive integer. Then $F_N(x) = \sum_1^N (+) x^k / N \leq \sum_1^N |x^k| / N \leq N^{1/2} / N = 1/N^{1/2}$.

In E_n we let N be any positive integer not less than N_1 and suppose that $N_r \leq N < N_{r+1}$. Then

$$\begin{aligned} |F_N(x) - \rho| \leq & \left| \sum_{\mu \geq r} (+) x^i \xi_k^{(\mu)} \Delta_k^{(\mu)} S / (\nu + \eta) S - V_{n-1} / S \right| \\ & + S / (\nu + \eta) S, \end{aligned}$$

where $0 \leq \eta < 1$. Hence, by (2),

$$\begin{aligned} |F_N(x) - \rho| \leq & \left| \nu \int_S (+) x^i \xi^i d\omega / (\nu + \eta) S - V_{n-1} / S \right| \\ & + S(1/2 + 1/4 + \cdots + 1/2^r) / (\nu + \eta) S + 1 / (\nu + \eta) \\ \leq & \left| \nu V_{n-1} / (\nu + \eta) S - V_{n-1} / S \right| + 2 / (\nu + \eta), \end{aligned}$$

and the conclusion follows.

THEOREM 9. $\sigma = \rho$.

THEOREM 10. There exist an uncountably infinite number of subseries $\sum b_j^*$ of the series $\sum b_k$ of the proof of Lemma 9.2 for which

$$\lim_N \left| \sum' b_j^* \right| / \sum_1^N \bar{B}_k = \max_T \limsup_N \left| \sum' b_j \right| / \sum_1^N \bar{B}_k = \rho = \sigma.$$

PROOF. In E_∞ each subseries $\sum b_j^*$ of $\sum b_k$ is a subseries $\sum b_j^*$ of the type required. (Compare the first part of the proof of Theorem 4.) In E_n the proof proceeds as in [1].

REMARKS. The formula of Theorem 4, which of course provides again the value $\rho = 1/\pi$ in E_2 which was derived in [1], also gives $\rho = 1/2$ in E_1 , which is correct, although $\inf_A \sup_S \left| \sum a_j' \right| / \sum A_k$ is attainable in E_1 . Since finally, as noted by A. E. Mayer [2], ρ as a function of n is asymptotically equal to $1/(2\pi n)^{1/2}$, we have an intuitive verification of the situation in E_∞ .

REFERENCES

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ON SETS OF CONSTANT WIDTH

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Sets of constant width in E^n (E^n will denote Euclidean n -space) form an important subset of the set of all convex sets in E^n . For example, the Hausdorff k measure ($0 < k \leq n$) of a subset of E^n is usually defined in terms of coverings of this set by arbitrary sets. However, because of the property that an arbitrary set is contained in a set of constant width of the same diameter, it is possible to refine this definition so that we need only consider coverings by sets of constant width. Unfortunately one cannot further refine the definition of Hausdorff measure, say to coverings by spheres, since it is not true that an arbitrary set is contained in a sphere of the same diameter. Thus in computing Hausdorff measures, the problem arises to find properties of sets of constant width that are analogous to those of spheres. It is the purpose of this paper to prove such a property; namely we establish a uniformity condition on sets of constant width (which incidentally does not hold for arbitrary convex sets). Roughly speaking, this uniformity condition states that parallel cross-sections of a set of constant width vary continuously and that the variation for sets of given diameter d can be characterized independent of the particular set. Though the motivation given above for theorems of this type arises from measure theory, I think that the results are interesting from a purely geometric point of view.

For the properties of convex sets and, in particular, sets of constant width, that will be used in this paper, the reader is referred to *Theorie der konvexen Körper* by J. Bonnesen and W. Fenchel, *Ergebnisse der Mathematik*, vol. 3, part 1, 1934. We shall in addition use the following notation:

As usual, the symbol \rightarrow will denote *implication* between sen-