FUNCTIONAL UNIFORMITIES

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This paper is concerned with uniform structures in the sense of André Weil [4]. In his proof that every Hausdorff uniform space is completely regular, Weil obtained a result which may be phrased as follows.

Weil’s Theorem. If $V$ is a uniformity (uniform structure) on the completely regular space $X$, given an “entourage” $V_a$ there exists a uniformly equicontinuous set $F$ of bounded real functions on $X$ and a $\delta > 0$ such that corresponding to each $P \in X$ there is an $f_P \in F$ with the property that $Q \in V_a(P)$ if $|f_P(P) - f_P(Q)| < \delta$.

This result suggests that every uniformity on $X$ can be defined by means of a family of sets of real bounded continuous functions, and that important uniformities on $X$ (such as precompact [1] ones) are definable by means of “nice” sets of such functions. These and related problems are investigated here.

A family $\mathcal{F}$ of sets of real functions will be said to separate a topological space $X$ if for each $P \in X$ and neighborhood $N(P)$ there exists an $F \in \mathcal{F}$ and a $\delta > 0$ such that if $|f(P) - f(Q)| < \delta$ for all $f \in F$, then $Q \in N(P)$.

Let $X$ be a completely regular space, let $\mathcal{F}$ be a family of sets of real continuous functions on $X$ with the following two properties:

(a) $\mathcal{F}$ separates $X$.
(b) Each $F \in \mathcal{F}$ is equicontinuous.

For each $F \in \mathcal{F}$ and real number $\delta > 0$, let $V_a = \{P, Q \in X \mid |f(P) - f(Q)| < \delta \text{ for all } f \in F\}$. Then these symmetric “entourages” $V_a$ satisfy the axioms for a uniformity on $X$. First, $\cap_a V_a = \Delta$, the diagonal in $X^2$; for if $(P, Q) \in V_a$ for every $a$, it follows that $f(P) = f(Q)$ for every $f$.

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1. Numbers in brackets refer to the list of references at the end of the paper.
2. A completely regular space is a Hausdorff space with the property that given any point $x \in X$ and closed set $K \subset X$ not containing $x$, there exists a real continuous function $f$ such that $f(x) = 1, f(y) = 0$ for all $y \in K$.
3. A set $F$ of real functions on a topological space $X$ is equicontinuous at $P \in X$ if given $\varepsilon > 0$ there exists a neighborhood $N(P)$ such that if $Q \in N(P)$, then $|f(Q) - f(P)| < \varepsilon$ for all $f \in F$. $F$ is equicontinuous if it is equicontinuous at every $P \in X$. If $X$ is a uniform space, $F$ is uniformly equicontinuous if given $\varepsilon > 0$ there exists an entourage $V_\varepsilon$ such that if $(P, Q) \in V_\varepsilon$ then $|f(Q) - f(P)| < \varepsilon$ for all $f \in F$. Note that if the members of an equicontinuous $F$ are uniformly continuous, it does not necessarily follow that $F$ is uniformly equicontinuous.
f \in F \in \mathcal{F}$, and by (a) this implies that $Q$ is contained in every neighborhood of $P$, which means that $Q = P$. Second, given $\alpha, \beta$ there exists $\gamma$ such that $V_\gamma \subset V_\alpha \cap V_\beta$; for by (b) there exist open sets $O$ and $O'$ in $X^2$, both containing $\Delta$, and respectively contained in $V_\alpha$ and $V_\beta$, hence $O \cap O' \subset V_\alpha \cap V_\beta$; but by (a), there is a $\gamma$ such that $V_\gamma \subset O \cap O'$. Third, given $\alpha$ there is a $\beta$ such that $V_\beta V_\beta \subset V_\alpha$; for if $\alpha = (F, \delta)$, we need only take $\beta = (F, \delta/2)$.

This uniformity on $X$ is compatible with the topology of $X$, for it induces a topology which by (a) is as fine as that of $X$, and by (b) is at most as fine as that of $X$.

Such a uniformity $V$, that is, any uniformity thus determined by a separating family of equicontinuous sets of real functions on $X$, will be called a functional uniformity on $X$. If all the members of every $F \in \mathcal{F}$ are bounded, we call $V$ a bounded functional uniformity on $X$.

**Theorem 1.** Every uniformity $V$ on a completely regular space $X$ is isomorphic to a bounded functional uniformity.

**Proof.** Let $\mathcal{F}$ be the family of all uniformly equicontinuous sets of real bounded functions on $V$. By Weil's Theorem, the functional uniformity determined by $\mathcal{F}$ is at least as fine as $V$; the uniform equicontinuity of the members of $\mathcal{F}$ implies that $V$ is at least as fine as the functional uniformity determined by $\mathcal{F}$.

**Corollary.** The finest uniformity on a completely regular space is the bounded functional uniformity determined by the family of all equicontinuous sets of real bounded functions.

We note in passing the following theorem:

**Theorem 2.** A completely regular space $X$ is metrizable if and only if it possesses a separating family consisting of a single equicontinuous set of real bounded functions.

**Proof.** If there exists such a set $F$, let $V$ be the functional uniformity determined by it, so that $V_\epsilon = \{P, Q \in X | |f(Q) - f(P)| < \epsilon \}$ for all $f \in F$. $V$ is clearly isomorphic to a first countable uniformity, hence [4] to a metric uniformity. Conversely, if $X$ is metrizable, it possesses a bounded metric $\rho(P, Q)$. For a fixed $P$, $\rho(P, Q)$ defines a bounded continuous function on $X$. The set of functions obtained by...
letting \( P \) range over \( X \) is equicontinuous (by the triangle inequality); it separates \( X \), because if \( P \in X \) and \( N(P) \) is a neighborhood of \( P \), by taking \( \delta \) so that the sphere neighborhood of \( P \) of radius \( \delta \) lies in \( N(P) \), we see that if \( |\rho(P, Q) - \rho(P, P)| < \delta \), then \( Q \in N(P) \).

If a separating family \( \mathcal{F} \) on a completely regular space \( X \) consists of finite sets only, the functional uniformity it determines will be called a weak functional uniformity. If in addition all the members of each \( F \in \mathcal{F} \) are bounded, it will be called a weak bounded functional uniformity.

**Lemma 1.** Let \( V \) be a uniformity on a completely regular \( X \). Suppose an “entourage” \( V_\alpha \) has the property that there exists a \( \delta > 0 \) and a finite (or infinite of cardinal number \( \mu \)) set \( F \) of real bounded continuous functions such that for each \( P \in X \) there is a finite set \( f_1, \ldots, f_m \in F \) with the property that \( |f_i(Q) - f_i(P)| < \delta \) for \( i = 1, \ldots, m \) implies \( Q \in V_\alpha(P) \). Then there exists a finite (or infinite of cardinal number \( \mu \)) set of points \( E \subset X \) such that \( \bigcup_{E \in E} V_\alpha(P) = X \).

**Proof.** First we prove this in the finite case. Suppose \( (P, Q) \in V_\alpha \) if \( |f(Q) - f(P)| < \delta \) for all \( f \in F \) (where \( F \) is finite). Let \( a = \sup_{X \in F} f(x) \), \( b = \inf_{x \in X} f(x) \). Let the closed interval \((a, b)\) be divided into a finite number, say \( r \), of closed subintervals \( I_1, \ldots, I_r \), each of length less than \( \delta \). Let \( p_1, \ldots, p_m \) be any ordered set of \( m \) positive integers (not necessarily distinct) each less than or equal to \( r \), and let \( E_{p_1, \ldots, p_m} = \{ P \in X \mid f_j(P) \in I_{p_j} \text{ for } j = 1, \ldots, m \} \). From each such set \( E_{p_1, \ldots, p_m} \) which is not empty choose a point, and order the resulting finite set \( P_1, \ldots, P_n \) (\( n \) is at most \( r^m \)). Let \( P \in X \); there is clearly an \( i \) such that \( |f_i(P) - f_i(P_i)| < \delta \) for \( j = 1, \ldots, m \). Hence \( (P, P_i) \in V_\alpha \). Thus \( \bigcup_{i=1}^n V_\alpha(P_i) = X \).

To complete the proof of the lemma, suppose \( V_\alpha \) has the property that for each \( P \in X \) there is a finite set \( F \subset F \) such that \( |f(Q) - f(P)| < \delta \) for all \( f \in F \) implies \( Q \in V_\alpha(P) \). For any finite set \( F \subset F \), let \( E_F = \{ P \in X \mid f(P) = F \} \). The set \( E_F \) is a uniform space with the property that if \( P, Q \in E_F \) and \( |f(Q) - f(P)| < \delta \) for all \( f \in F \), then \( Q \in V_\alpha(P) \cap E_F \). Hence, by the proof just given for the finite case, there exists a finite set \( [P] \in E_F \) such that \( \bigcup_{F \in E\cap F} V_\alpha(P) \cap E_F = E_F \). Since the set of finite subsets \( F \) of \( F \) has the same cardinal number as \( F \), the set of sets \( E_F \) has that same cardinal number. Furthermore, \( \bigcup_{E \in E} V_\alpha(P) = X \). Hence the set \( E = \bigcup_{[P] \in E} [P] \) has the property \( X = \bigcup_{F \in E} V_\alpha(P) \).

**Lemma 2.** Let \( V \) be a uniformity on a completely regular space \( X \). Suppose for every \( \alpha \) there exists a subset \( E \) of \( X \) which is finite (or infinite of cardinal number \( \mu \)) such that \( \bigcup_{E \in E} V_\alpha(P) = X \). Then for each \( \alpha \)
there exists a $\delta > 0$ and a finite set $F$ (or an infinite set $F$ of cardinal number $\mu$) of real bounded functions which is uniformly equicontinuous and which has the property that for each $P \in X$ there is an $f_P \in F$ such that if $|f_P(Q) - f_P(P)| < \delta$, then $Q \in V_{\alpha}(P)$.

**Proof.** Let $\alpha$ be given. Choose $\beta$ so that $V_{\beta} \subset V_{\alpha}$. According to Weil's Theorem, there is a $\delta > 0$ and a uniformly equicontinuous set $F'$ of real bounded functions such that given $P \in X$ there is an $f_P \in F'$ with the property that $|f_P(Q) - f_P(P)| < 2\delta$ implies $Q \in V_{\beta}(P)$. Let $\gamma$ be such that $V_{\gamma} \subset V_{\beta}$, and $(P, Q) \in V_{\gamma}$ implies $|f_P(Q) - f_P(P)| < \delta$; this is possible by the uniform equicontinuity of $F$. Let $E \subset X$ be a finite set (or a set of cardinal number $\mu$) such that $U_{P \in E} V_{\gamma}(P) = X$. Then the set $F$ of functions given by $f_P$ as $P$ ranges over $E$ has the property required. For let $P \in X$, and let $\bar{P} \in E$ be such that $P \in V_{\gamma}(\bar{P})$. Then $|f_{\bar{P}}(P) - f_{\bar{P}}(\bar{P})| < \delta$. If $|f_{\bar{P}}(Q) - f_{\bar{P}}(P)| < \delta$, it follows that $|f_{\bar{P}}(Q) - f_{\bar{P}}(\bar{P})| < 2\delta$. Hence $Q \in V_{\beta}(P)$. But since $(\bar{P}, P) \in V_{\beta}$, it follows that $Q \in V_{\alpha}(P)$.

**Theorem 3.** The following conditions on a uniformity $V$ on a completely regular space $X$ are equivalent.

I. For each $\alpha$, there exists a finite set (or an infinite set of cardinal number $\mu$) $E \subset X$ such that $U_{P \in E} V_{\alpha}(P) = X$.

II. For each $\alpha$, there exists a $\delta > 0$ and a uniformly equicontinuous set of real bounded functions which is finite (or infinite of cardinal number $\mu$) such that for each $P \in X$ there is a finite set $F_P \subset F$ such that $Q \in V_{\alpha}(P)$ whenever $|f(Q) - f(P)| < \delta$ for all $f \in F_P$.

**Proof.** By Lemma 2, I $\Rightarrow$ II. By Lemma 1, II $\Rightarrow$ I.

**Theorem 4.** The precompact uniformities are identical with the weak bounded functional uniformities; more precisely, every precompact uniformity is isomorphic to a weak bounded functional uniformity, and every weak bounded functional uniformity is precompact.

**Proof.** If $V$ is precompact, it satisfies condition I of Theorem 3 in the finite case [1], hence condition II. Therefore the weak bounded functional uniformity determined on $X$ by the family $\mathcal{F}$ of all finite sets of real bounded uniformly continuous functions on $V$ is at least as fine as $V$; it is not finer than $V$ since every finite set of uniformly continuous functions is uniformly equicontinuous. Conversely, if $V$ is a weak bounded functional uniformity, it satisfies condition II of Theorem 3 in the finite case, hence condition I in the finite case. Hence [1] $V$ is precompact.

Doss [2] has shown that a completely regular space has the prop-
Every real continuous function attains its maximum (or equivalently every real continuous function is bounded) if and only if every uniformity on $V$ is precompact; Hewitt [3] calls such a space pseudo-compact. Now if a uniformity on a completely regular space $X$ is precompact, so is every less fine uniformity; hence a completely regular $X$ is pseudo-compact if and only if the finest uniformity on $X$ is precompact. Using the corollary to Theorem 1, and Theorem 4, we obtain the following theorem.

**Theorem 5.** A completely regular space $X$ is pseudo-compact if and only if corresponding to every equicontinuous set $F$ of real bounded functions on $X$ and $\delta > 0$, there exists a finite set $f_1, \ldots, f_m$ of real bounded continuous functions and a $\delta' > 0$ such that $|f_i(Q) - f_i(P)| < \delta'$ for $i = 1, \ldots, m$ implies $|f(Q) - f(P)| < \delta$ for all $f \in F$.

Another consequence of Theorem 4 is the following.

**Theorem 6.** Given any uniformity $V$ on a completely regular space $X$, there exists a precompact uniformity $V'$ on $X$ which is at most as fine as $V$ and has the same set of real bounded uniformly continuous functions as $V$. The uniform completion of $V'$ furnishes a compactification $\overline{X}$ of $X$ such that the only real continuous functions on $X$ which can be extended to $\overline{X}$ are those which are uniformly continuous on $V$.

For the weak bounded functional uniformity $V'$ on $X$ determined by the family $\mathcal{F}$ of all finite sets of real bounded uniformly continuous functions on $V$ is at most as fine as $V$, since the family $\mathcal{F}$ is contained in the family of all uniformly equicontinuous sets of real bounded functions on $V$, and the uniformity determined by the latter is isomorphic to $V$. By definition of $V'$, every real bounded function uniformly continuous on $V$ is uniformly continuous on $V'$, and the converse holds because $V'$ is at most as fine as $V$.

**Theorem 7.** If $V$ is a weak functional uniformity on a completely regular $X$, for each $\alpha$ there exists a countable set $E \subset X$ such that $\bigcup_{P \in E} V_\alpha(P) = X$.

**Proof.** Let $\mathcal{F}$ be the family of countable sets of continuous functions on $X$ which determines $V$. Given $\alpha$, let $F$, $\delta$ be such that $V_\alpha = \{ P, Q \in X \mid |f(P) - f(Q)| < \delta \text{ for all } f \in F \}$. For each positive integer $n$, let $E_n = \{ P \in X \mid -n \leq f(P) \leq n \text{ for all } f \in F \}$. Each $E_n$ is a uniform space satisfying the hypotheses of Lemma 1 with the given $\alpha$, $\delta$, $F$. Hence a finite number of $V_\alpha(P)$ ($P \in E_n$) covers $E_n$. But $X = \bigcup_{n} E_n$. Hence a countable number of $V_\alpha(P)$ ($P \in X$) covers $X$.

It follows from Theorems 7 and 3 that every weak functional
uniformity on $X$ satisfies condition II of Theorem 3 with $\mu = \mathbb{N}_\delta$. This could also be proved directly as follows. Let $V_a = \{P, Q \in X \mid f_i(Q) - f_i(P) < \delta \text{ for } i = 1, \ldots, m\}$. Let $F_i = f_i/(|f_i| + 1)$ and consider the countable set of bounded functions $nF_i$ ($n$ ranges over the positive integers). Let $P \in X$. Since $f_i = F_i/(1 - |F_i|)$, it is clear that there exists $\delta'$ such that if $|F_i(Q) - F_i(P)| < \delta'$, then $|f_i(Q) - f_i(P)| < \delta$. Choosing $n$ so large that $n\delta' > \delta$ for all $i = 1, \ldots, m$, we see that if $|nF_i(Q) - nF_i(P)| < \delta$, then $|f_i(Q) - f_i(P)| < \delta$.

References