1. Introduction. The real functions

\[ x_j = x_j(u, v) = x_j(z), \quad j = 1, 2, 3, \]

defined and continuous in the unit disc \( D: |z| < 1 \) will be said to define a surface \( S \). In this note we shall study the equation

\[ \sum_{j=1}^{3} \left[ \int_{D(z, r)} (\xi - z) x_j(\xi) d\xi d\eta \right] = 0, \quad \xi = \xi + i\eta, \]

where \( D(z, r) \) denotes the closed disc in \( D \) with center \( z \) and radius \( r \). The equation (2) may be considered to be a generalization of a familiar equation of Fédéroff [3, p. 512].

If the first partial derivatives of the functions (1) are continuous and satisfy

\[ E(u, v) = G(u, v), \quad F(u, v) = 0 \]

in \( D \), where

\[ E(u, v) = \sum_{i=1}^{3} \left( \frac{\partial x_i}{\partial u} \right)^2, \quad F(u, v) = \sum_{i=1}^{3} \left( \frac{\partial x_i}{\partial u} \frac{\partial x_i}{\partial v} \right), \]

\[ G(u, v) = \sum_{i=1}^{3} \left( \frac{\partial x_i}{\partial v} \right)^2 \]

are the coefficients of the first fundamental differential quadratic form of \( S \), then the parameters \( u, v \) are said to be isothermic parameters, and \( S \) is said to be given in isothermic representation. If (3) holds, then the map of \( D \) on \( S \) is conformal except where \( E = G = 0 \). If \( EG - F^2 \neq 0 \) in \( D \), then we say \( S \) is regular. From (4) it follows that (3) may be written in the form

\[ \sum_{j=1}^{3} (\lambda x_j)^2 = 0, \]

where \( \lambda = \partial / \partial u + i\partial / \partial v \) is a differential operator.
If the functions (1) are harmonic and satisfy (5) in \( \mathcal{D} \), then they have been called a *triple of conjugate harmonic functions* [1]. In terms of these triples, a familiar theorem of Weierstrass may be stated as follows.

**Theorem 1.** A necessary and sufficient condition that the functions (1), defined in \( \mathcal{D} \), be the coordinate functions of a minimal surface in isothermic representation is that they constitute a triple of conjugate harmonic functions.

2. **Principal results.** We shall make use of the following result.

**Lemma.** If the functions (1) are not identically constant in \( \mathcal{D} \), if they have continuous partial derivatives of the third order in \( \mathcal{D} \), and if they are the coordinate functions of a regular surface \( S \), then a necessary and sufficient condition that they map \( \mathcal{D} \) isothermically on a surface that lies on a sphere \( S \) of finite non-null radius is that

\[
\sum_{j=1}^{3} \left[ \iint_{D(z_0,r)} (\xi - z) x_j(\xi) d\xi d\eta \right]^2 = o(r^8)
\]

hold at each point \( z \) in \( \mathcal{D} \), and that

\[
\sum_{j=1}^{3} \left[ \iint_{D(z_0,r)} (\xi - z_0) x_j(\xi) d\xi d\eta \right]^2 \neq o(r^8)
\]

hold at some point \( z_0 \) of \( \mathcal{D} \).

**Necessity.** If the functions (1) map \( \mathcal{D} \) isothermically on a spherical surface \( S \), then it follows from the formulas of Gauss [5, p. 359] that the functions (1) have continuous partial derivatives of all orders in \( \mathcal{D} \). Hence, if we use Taylor expansions, we find

\[
\sum_{j=1}^{3} \left[ \iint_{D(z_0,r)} (\xi - z) x_j(\xi) d\xi d\eta \right]^2 = \frac{\pi^2 r^4}{16} \sum_{j=1}^{3} (\lambda x_j)^2 + \frac{\pi^2 r^6}{96} \sum_{j=1}^{3} \lambda x_j \cdot \Delta \lambda x_j
\]

\[
+ \frac{\pi^2 r^4}{9216} \sum_{j=1}^{3} (3\lambda x_j \cdot \Delta^2 \lambda x_j + 4(\Delta \lambda x_j)^2) + o(r^8)
\]

where \( \Delta = \partial^2 / \partial u^2 + \partial^2 / \partial v^2 \) is a differential operator. Moreover it is also known that for this representation of \( S \),

\[
\sum_{j=1}^{3} \lambda x_j \cdot \Delta \lambda x_j = 0
\]
holds throughout $\mathcal{O}$ [5, p. 361]. From (5), (8), and (9) it follows that (6) holds throughout $\mathcal{O}$.

To show that there exist points at which (7) holds, we shall use a counterpositive proof. Suppose that

$$
\sum_{j=0}^{3} \left[ \int_{D(z, r)} (\xi - z) x_j(\xi) d\xi d\eta \right]^2 = o(r^g)
$$

holds at each point of $\mathcal{D}$. Then it follows from (8) that

$$
\sum_{j=1}^{3} (3\lambda x_j \Delta^2 \lambda x_j + 4(\Delta \lambda x_j)^2) = 0
$$

holds throughout $\mathcal{D}$. But for $S$ it is known [5, p. 359] that

$$
\sum_{j=1}^{3} \lambda x_j \Delta^2 \lambda x_j = -4\alpha^2 E \lambda^2 E, \quad \sum_{j=1}^{3} (\Delta \lambda x_j)^2 = 4\alpha^2 (\lambda E)^2, \quad \alpha \neq 0,
$$

where $\alpha$ is a real constant. From (11) and (12) we obtain

$$
3E\lambda^4 E - 4(\lambda E)^2 = 0.
$$

Now we want to show the existence of an open subset $\mathcal{D}^*$ of $\mathcal{D}$ where $\lambda E \neq 0$ and $E \neq 0$. First, if $E$ were identically constant in $\mathcal{D}$, then $S$ would be a plane surface, contrary to hypothesis. Hence there is an open subset $\mathcal{D}_1$ or $\mathcal{D}$ where $\lambda E \neq 0$. Second, if $E = 0$ throughout $\mathcal{D}_1$ then $\lambda E = 0$ there. Hence there is an open subset of $\mathcal{D}_1$ where $\lambda E \neq 0$ and $E \neq 0$, and this subset is denoted by $\mathcal{D}^*$.

For $z$ in $\mathcal{D}^*$, (13) yields $3\lambda \log (\lambda E) = 4\lambda (\log E)$, and hence

$$
\lambda E = E^{4/3}\Phi(z),
$$

where

$$
\lambda \Phi(z) = 0.
$$

From the imaginary part of (13) we obtain

$$
3 \frac{E_{uv}}{E_u} du = 4 \frac{E_v}{E} dv, \quad 3 \frac{E_{uv}}{E_v} = 4 \frac{E_u}{E} dv,
$$

and hence

$$
E_u = E^{4/3}\Phi_1(u), \quad E_v = E^{4/3}\Phi_2(v),
$$

where $\Phi_1(u)$ and $\Phi_2(v)$ are real functions of $u$ and $v$, respectively. From (14), (15), and (16) we obtain

$$
\Phi(z) = \Phi_1(u) + i\Phi_2(v),
$$
which, by (15), is an analytic function for \( z \) in \( \mathcal{D}^* \). If we use the Cauchy-Riemann equations for (17), then we find

\[
\Phi_1(u) = 2a_0u + a_1, \quad \Phi_2(v) = 2a_0v + a_2,
\]

where \( a_0, a_1, a_2 \) are real constants. Therefore (14) yields

\[
\frac{E_u}{E_{1/3}} du = (2a_0u + a_1)du, \quad \frac{E_v}{E_{1/3}} dv = (2a_0v + a_2)dv,
\]

from which we obtain

\[
E = -\frac{27}{[a_0(u^2 + v^2) + a_1u + a_2v + a_3]^3},
\]

where \( a_3 \) is another real constant. But it is well known that the Gaussian curvature of \( S \) is given by [2]

\[
K = -\frac{1}{2E} \Delta (\log E).
\]

From (18) and (19), we find

\[
K = \frac{a_1^2 + a_2^2 - 4a_0a_3}{18} \left[ a_0(u^2 + v^2) + a_1u + a_2v + a_3 \right],
\]

which holds throughout \( \mathcal{D}^* \). But \( S \) is on the sphere \( \mathbb{S} \), so that (20) must be identically constant in \( \mathcal{D}^* \). Hence \( a_0 = a_1 = a_2 = 0 \) and therefore \( K = 0 \); therefore \( \mathcal{D}^* \) is mapped on a spherical surface with vanishing curvature. But \( \mathbb{S} \) has finite radius, so that we have been led to a contradiction by assuming (10) to hold throughout \( \mathcal{D} \). Hence (7) holds for at least one point of, and hence in an open subset of, \( \mathcal{D} \).

**Sufficiency.** By the use of finite Taylor expansions we obtain

\[
\sum_{j=1}^{3} \left[ \int_{D(x,r)} (\xi - x_j) \frac{d\xi d\eta}{D(x,r)} \right]^2 = \frac{\pi^2 r^4}{16} \sum_{j=1}^{3} (\lambda x_j)^2 + \frac{\pi^2 r^6}{96} \sum_{j=1}^{3} \lambda x_j \Delta \lambda x_j + o(r^6).
\]

From (6) and (21) we obtain (5) and (9). If we operate on (5) with the operator \( \lambda \), we obtain

\[
\sum_{j=1}^{3} \lambda x_j \Delta x_j = 0.
\]

Operating on (22) with \( \lambda \), and applying (9) to the result, we obtain
\begin{align*}
\sum_{j=1}^{3} \lambda^2 x_j^2 \Delta x_j &= 0. \\
\text{There are four real, linear, homogeneous equations in } \Delta x_j, j=1, 2, 3, \\
\text{implied by (22) and (23):} & \sum_{j=1}^{3} x_{j,u} \Delta x_j = 0, \\
& \sum_{j=1}^{3} x_{j,v} \Delta x_j = 0, \\
& \sum_{j=1}^{3} (x_{j,uu} - x_{j,uv}) \Delta x_j = 0, \\
& \sum_{j=1}^{3} x_{j,uv} \Delta x_j = 0. \\
\text{One solution to the system (24) is} & \Delta x_j = 0, \quad j = 1, 2, 3.
\end{align*}

In this case the functions (1) are harmonic in } \mathcal{D}, \text{ and hence (8) yields (10) for each } D(z, r) \text{ in } \mathcal{D}. \text{ This contradicts (7). Hence the set } \mathcal{D}^{**},
\begin{align*}
\mathcal{D}^{**} &= \left\{ z \in \mathcal{D}; \sum_{j=1}^{3} (\Delta x_j)^2 \neq 0; (24) \text{ holds} \right\},
\end{align*}

is a non-null open subset of } \mathcal{D}.

For } z \text{ in } \mathcal{D}^{**}, \text{ the system (24) has a nontrivial solution. Therefore the rank of the matrix}
\begin{align*}
\begin{vmatrix}
x_{1,u} & x_{2,u} & x_{3,u} \\
x_{1,v} & x_{2,v} & x_{3,v} \\
x_{1,uu} - x_{1,uv} & x_{2,uu} - x_{2,uv} & x_{3,uu} - x_{3,uv} \\
x_{1,uv} & x_{2,uv} & x_{3,uv}
\end{vmatrix}
\end{align*}
is less than three. Hence it follows from the definitions of the coefficients } e, f, g, \text{ of the second fundamental differential quadratic form of } S, \text{ that}
\begin{align*}
e = g, \quad f = 0,
\end{align*}
holds in } \mathcal{D}^{**}. \text{ Hence it follows from (5) and (26) that the functions (1) map } \mathcal{D}^{**} \text{ isothermically on a spherical surface } S^{**}. \text{ If } S^{**} \text{ were either a plane or a point, then (25) would hold in } \mathcal{D}^{**}; \text{ hence } S^{**} \text{ lies on a sphere of finite non-null radius.}

Now consider the subset } \mathcal{D} - \mathcal{D}^{**} \text{ of } \mathcal{D}, \text{ and let } z_0 = \alpha + i\beta \text{ be a point of } \mathcal{D} - \mathcal{D}^{**}. \text{ If } z_0 \text{ is a frontier point of } \mathcal{D} - \mathcal{D}^{**}, \text{ then a simple continuity argument shows that the functions (1) map } z_0 \text{ on the boundary of } S^{**}, \text{ lying on the same sphere containing } S^{**}; \text{ moreover, since } S \text{ is regular, we know that } E(\alpha, \beta) \neq 0, \text{ so that the con-}
tinuity of the Gaussian curvature shows that $K(u_0, v_0) > 0$. Now if $D - D^{**}$ has a component $C$ with non-null interior, then the functions (1) map this interior on a minimal surface $\mathcal{M}$, such that the frontier of $C$ is mapped on the boundary of $\mathcal{M}$. Since a minimal surface is a surface of nonpositive Gaussian curvature, it follows by a continuity argument again that the Gaussian curvature at points of the boundary of $\mathcal{M}$ satisfy $K(u_0, v_0) \leq 0$. But at the beginning of this paragraph it was pointed out that $K(u_0, v_0) > 0$ at the points corresponding to frontier points of $D - D^{**}$. Since $S^{**}$ and $\mathcal{M}$ are merely parts of the regular surface $S$, it follows from this contradiction that the set $D - D^{**}$ has no component with a non-null interior, and hence all points $D - D^{**}$ are mapped on the boundary of $S^{**}$. Therefore the functions (1) map all of $D$ isothermically on a spherical surface, lying on a sphere with finite non-null radius.

The following result does not appear to demand the regularity of the surfaces involved.

**Theorem 2.** If the functions (1) have continuous partial derivatives of the third order in $D$, then a necessary and sufficient condition that they be the coordinate functions of a minimal surface in isothermic representation is that

\[
\sum_{j=1}^{3} \left[ \int_{D(z, r)} (\xi - z) x_j(\xi) d\xi d\eta \right]^2 = 0
\]

hold for each $D(z, r)$ in $D$.

**Necessity.** If the functions (1) map $D$ isothermically on a minimal surface, then it follows from Theorem 1 that (5) and (25) hold, and hence, by the use of Fourier expansions, it follows that (27) holds for each $D(z, r)$ in $D$.

**Sufficiency.** If (27) holds, then (5), (22), and (23) follow from (21). Hence we obtain the system (24). Now consider the set $D^{**}$ defined in the proof of the lemma. As in the proof, the functions (1) map $D^{**}$ isothermically on a surface $S^{**}$ that lies on a sphere with finite non-null radius. Hence the functions (1) are analytic in the real variables $u, v$ [5, p. 358]. Therefore we may use (8) to obtain (11). Hence we find that (18) holds in $D^{**}$, and consequently, as in the proof of the necessity part of the lemma, the Gaussian curvature of $S^{**}$ is identically zero. Therefore the spherical surface $S^{**}$ is a plane surface, so that (25) holds in $D^{**}$. From this contradiction we conclude that the open set $D^{**}$ contains no interior points, hence is a null set. Therefore (5) and (25) hold throughout $D$, and hence $S$ is a minimal surface given in isothermic representation by the functions (1).
Corollary. If the functions (1) have continuous partial derivatives of the third order in $D$, then a necessary and sufficient condition that they be the coordinate functions of a minimal surface in isothermic representation is that (10) hold at each point of $D$.

3. Characterization of plane isothermic maps. We take this occasion to offer a simplification of a proof of a characterization of plane isothermic maps, a result which may be considered to be a generalization to space of the Cauchy and Morera theorems. The simplification consists in deriving some fundamental identities without the use of conformal mapping and schlicht functions [5, pp. 368–370].

Theorem 3. If the functions (1) have continuous partial derivatives of the third order in $D$, then a necessary and sufficient condition that they be the coordinate functions of a plane surface in isothermic representation is that

$$
\sum_{j=1}^{3} \left[ \int_{\gamma} x_j(\xi) d\xi \right]^2 = 0
$$

hold for each closed rectifiable Jordan curve $\gamma$ in $D$.

Proof. In the earlier proof it was shown that (28) is a necessary condition that the functions (1) map $D$ isothermically on a plane surface. It was also shown that if (28) holds for all closed rectifiable Jordan curves $\gamma$ in $D$, then the functions (1) constitute a triple of conjugate harmonic functions in $D$ and hence may be written in the form

$$
x_j(z) = F_j(z) + \overline{F_j(z)}, \quad j = 1, 2, 3,
$$

where $F_j(z)$ is analytic in $D$ and where $\overline{F_j(z)}$ is its complex conjugate.

Now let $z = u + iv$ be an arbitrary point in $D$, and write

$$
F_j(\xi) = \sum_{n=0}^{\infty} a_{j,n} (\xi - z)^n, \quad \overline{F_j(\xi)} = \sum_{n=0}^{\infty} \bar{a}_{j,n} (\overline{\xi - z})^n.
$$

Now consider the family of triangles $\gamma(r, \alpha)$, with vertices $z, z + re^{i\alpha}, z + re^{-i\alpha}$. Then a direct computation, using (28) and (29), yields

$$
\sum_{j=1}^{3} \left[ \int_{\gamma(r, \alpha)} x_j(\xi) d\xi \right]^2
$$

$$
= - 4 \cos^2 \alpha \sum_{n=0}^{\infty} r^{n+1} \sum_{k=0}^{n} \sin k\alpha \sin (n - k)\alpha \left[ \sum_{j=1}^{3} d_{j,k} d_{j,n-k} \right] = 0,
$$

which must hold for $\gamma(r, \alpha)$ in $D$. Since (30) holds for all sufficiently
small \( r \), it follows that

\[
\sum_{k=0}^{n} \sin k\alpha \sin (n - k)\alpha \left[ \sum_{i=1}^{3} \bar{a}_{i,k} \bar{a}_{i,n-k} \right] = 0, \quad n \geq 0,
\]

holds for all \( \alpha \). From (31) we obtain

\[
\sum_{k=1}^{n-1} \left[ \cos (n - 2k)\alpha - \cos n\alpha \right] \left[ \sum_{i=1}^{3} \bar{a}_{i,k} \bar{a}_{i,n-k} \right] = 0, \quad n \geq 2,
\]

which must hold for all \( \alpha \). Now let \( p \) be a fixed integer, \( 1 \leq p \leq n - 1 \). If we multiply both members of (32) by \( \cos(n - 2p)\alpha \) and then integrate the result over the interval \((0, 2\pi)\), we obtain

\[
\bar{a}_{n-p}, \bar{a}_{n+p} = 0, \quad 1 \leq p \leq n - 1, \quad n \geq 2.
\]

Now (33) are the identities that had to be proved in order to prove that the functions (1) map \( \mathcal{D} \) isothermically on a plane surface.

4. Conclusion. The results contained in this note are similar to those obtained before \([5]\). The one outstanding point of difference is the lack of a characterization of isothermic plane maps by means of an equation similar to (5). It would be interesting to obtain such a characterization.

Bibliography


University of Michigan