ON A THEOREM OF NICOLESCO AND GENERALIZED LAPLACE OPERATORS

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1. Let D be a domain in the n-dimensional Euclidean space and U be a continuous function defined in D. Denote by

$$\mu_0(U;R) \equiv \mu_0(U;P,R) = \frac{1}{\sigma_n(R)}\int_{S_n(P,R)} U(P')d\sigma_{P'} \tag{1.1}$$

the spherical mean of U on the sphere $S_n(P, R)$, lying entirely in D, with center P and radius R, where $\sigma_n(R)$ is the surface volume of $S_n(P, R)$ and $d\sigma_{P'}$ is its volume element. Write $\mu_k(U;R) \equiv \mu_k(U;P,R) = (n/R^n)\int_0^R t^{n-1}\mu_{k-1}(U;t)dt$ for $k=1, 2, \cdots$.

Blaschke [1][2] proved that a necessary and sufficient condition for a continuous function $U(P)$ to be harmonic in D is that

$$\lim_{R\to 0}\frac{1}{R^2}\{\mu_0(U;P,R) - U(P)\} = 0 \tag{1.2}$$

holds for every point P in D. Nicolesco [4] (also [3, p. 10]) extended Blaschke's theorem as follows.

A necessary and sufficient condition for a continuous function $U(P)$ to be harmonic of order p in D is that, for every point P in D,

$$\lim_{R\to 0}\frac{1}{R^{2p}}\left\{\frac{V[\mu, n, p]}{V[1, n, p]} - U(P)\right\} = 0, \tag{1.3}$$

where $V[\mu, n, p]$ and $V[1, n, p]$ denote the two determinants

$$\begin{vmatrix} \mu_0(U;P,R) & 1 & \cdots & 1 \\ \mu_1(U;P,R) & \dfrac{n}{n+2} & \cdots & \dfrac{n}{n+2p-2} \\ \mu_2(U;P,R) & \dfrac{n^2}{(n+2)^2} & \cdots & \dfrac{n^2}{(n+2p-2)^2} \\ \vdots & \vdots & & \vdots \\ \mu_{p-1}(U;P,R) & \dfrac{n^{p-1}}{(n+2)^{p-1}} & \cdots & \dfrac{n^{p-1}}{(n+2p-2)^{p-1}} \end{vmatrix}$$

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[2] Numbers in brackets refer to the references at the end of the paper.

and

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & \dfrac{n}{n+2} & \cdots & \dfrac{n}{n+2p-2} \\ 1 & \dfrac{n^2}{(n+2)^2} & \cdots & \dfrac{n^2}{(n+2p-2)^2} \\ \vdots & \vdots & & \vdots \\ 1 & \dfrac{n^{p-1}}{(n+2)^{p-1}} & \cdots & \dfrac{n^{p-1}}{(n+2p-2)^{p-1}} \end{vmatrix}$$

respectively.

That condition (1.3) in Nicolesco's theorem is necessary follows from his own extension [5] of Gauss' theorem. But the condition is no longer a sufficient one even in the case that U belongs to C', that is, that U is continuous together with its first partial derivatives. This can be easily seen from the counter example given below.

2. Let us consider the simple case $n=2$ and $p=2$. In this case, (1.3) reads

$$\lim_{R\to 0} \frac{1}{R^4}\{U(P) + \mu_0(U; R) - 2\mu_1(U; R)\} = 0, \tag{2.1}$$

where

$$\mu_0(U; R) = \frac{1}{2\pi}\int_0^{2\pi} U(x + R\cos\theta,\ y + R\sin\theta)d\theta$$

and

$$\mu_1(U: R) = \frac{2}{R^2}\int_0^R t\mu_0(U; t)dt.$$

Let us define

$$U(P) \equiv U(x, y) = \begin{cases} x^2 + y^2 & (-1 < x < 1,\ 0 \leqq y < 1), \\ x^2 - y^2 & (-1 < x < 1,\ -1 < y < 0). \end{cases}$$

Obviously U belongs to C' throughout in the domain

$$D:\ -1 < x < 1,\ -1 < y < 1.$$

Moreover, since U is biharmonic both in the upper half square

$$-1 < x < 1, \qquad 0 < y < 1$$

and in the lower half square

$$-1 < x < 1, \qquad -1 < y < 0,$$

we have only to verify condition (2.1) for

$$-1 < x < 1, \qquad y = 0.$$

Let $P = (x, 0)$ be a point on the above segment and $\delta > 0$ be chosen so small that, for $0 < R \leqq \delta$,

$$-1 < x - R < x + R < 1.$$

Then it is easy to see that

$$\begin{aligned} \mu_0(U; P, R) &= \frac{1}{2\pi}\int_0^{2\pi} (x + R\cos\theta)^2 d\theta + \frac{1}{2\pi}\int_0^{\pi} (R\sin\theta)^2 d\theta \\ &\quad - \frac{1}{2\pi}\int_{\pi}^{2\pi} (R\sin\theta)^2 d\theta \\ &= x^2 + R^2/2 \qquad\qquad (0 < R \leqq \delta), \end{aligned}$$

and

$$\mu_1(U; P, R) = x^2 + R^2/4 \qquad\qquad (0 < R \leqq \delta).$$

Therefore (2.1) holds throughout in D. But evidently U is not biharmonic in D.

It should be noticed that, in the above example, the relation (2.1) does not hold uniformly on each closed subset of D which contains points of the segment $-1 < x < 1$ on the x-axis. It was pointed out to me by a referee that the following theorem, a slightly weaker form of Nicolesco's theorem, can be proved:

If U is continuous in a domain D, and if the relation (2.1) *holds uniformly on each closed subset of D, then $U(P)$ is biharmonic in D.*

The proof which the referee sketched to me is based on the fact that, if a function U belongs to C'''', then the relation (2.1) is equivalent to $\Delta^2 U = 0$. Here Δ^2 is the ordinary iterated Laplace operator.

We shall also mention here that the original proof given by Nicolesco [4] breaks down at the formula (18) on p. 241, since a factor $1/r^{2p}$ should not be neglected at the left-hand side of this formula.

3. Let us write

$$(3.1) \qquad \nabla^2 U = \lim_{R\to 0} \frac{32}{R^4}\left\{\mu_0(U; 2^{1/2}R) - 2\mu_0(U; R) + U(P)\right\}$$

and

$$(3.2)\qquad \tilde{\nabla}^2 U = \lim_{R\to 0} \frac{192}{R^4}\left\{U(P) + \mu_0(U; R) - 2\mu_1(U; R)\right\}.$$

If $U(x, y)$ belongs to C'''', then

$$\nabla^2 U = \tilde{\nabla}^2 U = \Delta^2 U(P).$$

This can be easily seen in virtue of the following theorem on mean-values due to Pizzetti [8] (for $n=3$, see Pizzetti [7], and for general n-dimensional case, see Nicolesco [5]):

If $U(x, y)$ belongs to $C^{(2m)}$, then

$$(3.3)\qquad \begin{aligned} \mu_0(U; P, R) = U(P) &+ \frac{R^2}{2^2}\Delta U(P) + \frac{R^4}{2^2 4^2}\Delta^2 U(P) + \cdots \\ &+ \frac{R^{2n-2}}{2^2\cdot 4^2 \cdots (2n-2)^2}\Delta^{2n-2}U(P) \\ &+ \frac{R^{2n}}{2^2\cdot 4^2\cdots (2n)^2}\Delta^{2n}U(P'), \end{aligned}$$

where P' is a certain point inside the circle $S_2(P, R)$.

Both the operators ∇^2 and $\tilde{\nabla}^2$ can be considered as generalized iterated Laplace operators. However, for the operator ∇^2, we can extend Blaschke's theorem as follows:

THEOREM 1. *If $U(x, y)$ belongs to C'' throughout in a domain D, and if*

$$(3.4)\qquad \nabla^2 U(x, y) = 0$$

for every point (x, y) in D, then $U(x, y)$ is biharmonic in D.

The same example given in §2 shows that the hypothesis that U belongs to C'' cannot be replaced by a weaker one that U belongs to C'.

We need the following lemma:

LEMMA. *Let U belong to C'' in a domain D. If*

$$\Delta U(P) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)U(x, y)$$

attains a maximum at a point Q in D, and if $\nabla^2 U$ exists at Q, then

$$\nabla^2 U(Q) \leqq 0.$$

PROOF. Suppose on the contrary that $\nabla^2 U(Q) > 0$, then

$$\mu_0(U; Q, 2^{1/2}R) - 2\mu_0(U; Q, R) + U(Q) > 0$$

for $0 < R < \delta$, say. This implies that

$$\frac{4}{(2^{1/2}R)^2}\{\mu_0(U; Q, (2^{1/2}R) - U(Q)\} > \frac{4}{R^2}\{\mu_0(U; Q, R) - U(Q)\} \qquad (0 < R \leqq \delta).$$

Thus for $0 < R < 2^{1/2}\delta$, we have

$$\begin{aligned} &\frac{4}{R^2}\{\mu_0(U; Q, R) - U(Q)\} \\ &\qquad > \frac{4}{(R/2^{1/2})^2}\left\{\mu_0\left(U; Q, \frac{R}{2^{1/2}}\right) - U(Q)\right\} > \cdots \\ &\qquad > \frac{4}{(R/(2^{1/2})^m)^2}\left\{\mu_0\left(U; Q, \frac{R}{(2^{1/2})^m}\right) - U(Q)\right\} > \cdots . \end{aligned}$$

Since U belongs to C'', the last member tends to $\Delta U(Q)$ as $m \to \infty$. Moreover by the mean value theorem we have

$$\frac{4}{R^2}\{\mu_0(U; Q, R) - U(Q)\} = \Delta U(Q') > \Delta U(Q)$$

where Q' is a certain point inside the circle $S_2(Q, R)$ and can be made arbitrarily near to Q as δ becomes small. This contradicts the hypothesis that $\Delta U(Q)$ is a maximum.

4. In virtue of the lemma, Theorem 1 can be proved by an argument similar to that in Blaschke [1] (see also Potts [9]). In fact, let $P = (x_0, y_0)$ be any point in D, and let $S_2(P, a)$ be a circle with center P and radius a, lying entirely in D. Then we have only to show that

$$\Delta^2 U(x, y) = 0$$

holds inside $S_2(P, a)$.

Let $c > 0$ be an arbitrary constant and

$$\begin{aligned} h(x, y) = \frac{c}{4}\left\{\frac{1}{12}\left[(x - x_0)^4 + (y - y_0)^4\right]\right. \\ \left. - \frac{a^2}{4}\left[(x - x_0)^2 + (y - y_0)^2\right]\right\}. \end{aligned}$$

Let $V(x, y)$ be a biharmonic function in $S_2(P, a)$ subject to the boundary condition that

$$\Delta V(x,y) = \Delta U(x,y)$$

on $S_2(P, a)$. Such a function V can be easily established (see, for example, Nicolesco [6]). Now consider the function

$$W = U - V + h$$

which belongs to C'' inside the circle $S_2(P, a)$. It can be easily seen that

$$(4.1) \quad \Delta W = \Delta U - \Delta V + \frac{c}{4}\{(x - x_0)^2 + (y - y_0)^2 - a^2\}$$

and

$$(4.2) \quad \nabla^2 W = c$$

throughout in $S_2(P, a)$. Moreover $\Delta W = 0$ on the circle $S_2(P, a)$. Since $c>0$, ΔW must be nonpositive inside $S_2(P, a)$. Otherwise ΔW would attain a positive maximum at a certain point Q inside $S_2(P, a)$, which would contradict (4.2) in virtue of the lemma. Thus we have

$$\Delta U - \Delta V \leqq \frac{c}{4} a^2$$

throughout in $S_2(P, a)$. Similarly, by putting $W' = V - U + h$, we have also

$$\Delta V - \Delta U \leqq \frac{c}{4} a^2.$$

But c can be made arbitrarily small; then we have

$$\Delta U = \Delta V$$

throughout in $S_2(P, a)$. Thus

$$\Delta^2 U = \Delta^2 V = 0$$

in $S_2(P, a)$.

5. Another characteristic property of polyharmonic functions has been given by Cioranesco [2]. In case $p=2$ and $n=2$, his theorem is as follows:

If $U(x, y)$ is analytic in a domain D and if for every point (x, y) in D

$$(5.1)\qquad \lim_{R\to 0}\frac{1}{R}\int_0^{2\pi}\left[\frac{\partial^3}{\partial R^3}U(x+R\cos\theta,\ y+R\sin\theta)\right]d\theta = 0,$$

then $U(x, y)$ is biharmonic in D.

The above theorem is an extension of a theorem of Saks [10] concerning harmonic functions. We shall deduce here the following strengthened form of Cioranesco's theorem:

THEOREM 2. *If $U(x, y)$ belongs to C''' in a domain D, and if* (5.1) *holds for every point in D, then $U(x, y)$ is biharmonic in D.*

PROOF. It can be easily seen in virtue of the law of limit of indetermination that (5.1) implies (3.4). Then Theorem 2 follows from Theorem 1. In fact, let us write

$$\frac{g(R)}{R^4} \equiv \frac{1}{R^4}\{\mu_0(U;\ 2^{1/2}R) - 2\mu_0(U;\ R) + U(P)\}.$$

Then $g(0)=0$. By observing that

$$\lim_{R\to 0}\frac{\partial}{\partial R}\mu_0(U;\ R) = U_x\int_0^{2\pi}\cos\theta d\theta + U_y\int_0^{2\pi}\sin\theta d\theta = 0,$$

we have $g'(0)=0$. Also we have

$$g''(0) = \Delta U(x, y) - \Delta U(x, y) = 0.$$

Thus by the law of mean,

$$\frac{g(R)}{R^4} = \frac{g'(\rho)}{4\rho^3} = \frac{g''(\sigma)}{12\sigma^2} = \frac{g'''(\tau)}{24\tau} \qquad (0 < \tau < \rho < \sigma < R).$$

Since

$$\lim_{R\to 0}\frac{1}{R}g'''(R) = 0$$

by (5.1), we have

$$\lim_{R\to 0}\frac{g(R)}{R^4} = 0.$$

This proves the theorem.

By the same reasoning we can also prove Theorem 2 directly. Let us consider now the quotient

$$\frac{f(R)}{R^3} \equiv \frac{1}{R^3}\left\{R\frac{\partial^2}{\partial R^2}\mu_0(U;\ R) - \frac{\partial}{\partial R}\mu_0(U;\ R)\right\}.$$

Since $f(0)=0$, then

$$\frac{f(R)}{R^3} = \left[\frac{R\partial^3\mu_0(U;\, R)/\partial R^3}{3R^2}\right]_{R=\rho,0<\rho<R} = \frac{\partial^3\mu_0(U;\,\rho)/\partial\rho^3}{3\rho}.$$

Thus by (5.1) we have

$$\text{(5.2)}\qquad \lim_{R\to 0}\frac{f(R)}{R^3} = \lim_{R\to 0}\frac{\partial^2\mu_0(U;\, R)/\partial R^2 - (1/R)(\partial\mu_0(U;\, R)/\partial R)}{R^2} = 0.$$

On the other hand, by observing that

$$\frac{\partial^2}{\partial t^2}U(x+t\cos\theta,\, y+t\sin\theta) + \frac{1}{t^2}\frac{\partial^2}{\partial\theta^2}U(x+t\cos\theta,\, y+t\sin\theta)$$

$$= \Delta U(x+t\cos\theta,\, y+t\sin\theta) - \frac{1}{t}\frac{\partial}{\partial t}U(x+t\cos\theta,\, y+t\sin\theta),$$

we have the equality

$$\text{(5.3)}\qquad \frac{\partial^2}{\partial t^2}\mu_0(U;\, t) = \mu_0(f;\, t) - \frac{1}{t}\frac{\partial}{\partial t}\mu_0(U;\, t),$$

where $f\equiv f(x,\, y)=\Delta U(xy)$. Thus (5.2) reads

$$\lim_{R\to 0}\frac{\mu_0(f;\, R) - (2/R)(\partial\mu_0(U;\, R)/\partial R)}{R^2} = 0,$$

or

$$\text{(5.4)}\qquad \lim_{R\to 0}\left\{\frac{\mu_0(f;\, R) - f(x,\, y)}{R^2} - \frac{(2/R)(\partial\mu_0(U;\, R)/\partial R) - f(x,\, y)}{R^2}\right\} = 0.$$

It can be easily seen that

$$\lim_{R\to 0}\frac{2}{R}\frac{\partial}{\partial R}\mu_0(U;\, R) = f(x,\, y).$$

Thus the second term of (5.4) is again a limit of indetermination. Thus we have

$$\lim_{R\to 0}\frac{(2/R)(\partial\mu_0(U;\, R)/\partial R) - f(x,\, y)}{R^2}$$

$$= \lim_{R\to 0}\frac{(2/\rho)(\partial^2\mu_0(U;\,\rho)/\partial\rho^2) - (2/\rho^2)(\partial\mu_0(U;\,\rho)/\partial\rho)}{2\rho}$$

$$(0<\rho=\rho(R)<R)$$

which is equal to zero in virtue of (5.2). Then (5.4) reads

$$\lim_{R\to 0} \frac{\mu_0(f;R) - f(x,y)}{R^2} = 0.$$

It follows from the theorem of Blaschke that

$$f(x,y) = \Delta U(x,y)$$

is harmonic, or $U(x, y)$ is biharmonic.

6. The generalized Laplace operator ∇^2 can be defined by (3.1) for any dimension $n \geqq 2$ apart from a constant factor depending on n. Theorem 1 and Theorem 2 obviously hold for the general n-dimensional case. Theorem 1 also can be extended to polyharmonic functions of order p provided that ∇^p, the generalized pth iterated Laplace operator, is properly defined. In this case, U should be supposed of the class $C^{(2p-2)}$, and Theorem 2 can then also be extended to the polyharmonic case.

In case $n=2$, the generalized 3rd and 4th iterated Laplace operator should be defined as follows

$$\nabla^3 U = \lim_{R\to 0} \frac{96}{R^6}\left\{\mu_0(2R) - 6\mu_0((2^{1/2})R) + 8\mu_0(R) - 3U(P)\right\}$$

$$(\mu_0(R) = \mu_0(U; P, R)),$$

$$\nabla^4 U = \lim_{R\to 0} \frac{768}{7\cdot R^8}\left\{\mu_0(2(2^{1/2})R) - 14\mu_0(2R) + 56\mu_0(2^{1/2})R) - 64\mu_0(R) + 21U(P)\right\}.$$

The verification of the validity of Theorem 1 and Theorem 2 for $n=2$, $p=3$ and 4 is then obvious.

It would be desirable to obtain the result that Theorem 1 also holds for the operator $\tilde{\nabla}^2$; such a result would be closer to Blaschke's result than the present Theorem 1. However, the difficulty that occurs in our argument is that the lemma used in the proof of Theorem 1 might be invalid for the operator $\tilde{\nabla}^2$.

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