ON A THEOREM OF R. MOUFANG

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A loop is a system with a binary operation, possessing a unit 1, and such that any two of the elements in the equation $xy=z$ uniquely determine the third. A *Moufang* loop [1, chap. 2][1] may be characterized by the identity $xy\cdot zx = (x\cdot yz)x$. The following theorem is due to R. Moufang [2].

THEOREM. *If $ab\cdot c=a\cdot bc$ for three elements a, b, c of a Moufang loop, the subloop generated by them is associative.*

We give a particularly simple proof for the commutative case. (This proof, although complete in itself, stems from the theory of autotopisms introduced in [1], which will be applied elsewhere to the noncommutative case.) Henceforth let G be a *commutative* Moufang loop. For each x in G define the permutation $R(x)$ by $yR(x) =yx$. The defining relation can be written in the two forms

$$(1) \qquad yx\cdot zx = (yz\cdot x)x, \qquad yR(x)\cdot zR(x) = (yz)R(x)^2.$$

If we take $z=x$ in (1), $yx\cdot xx=(yx\cdot x)x$. If we replace yx by y,

$$(2) \qquad y\cdot xx = yx\cdot x.$$

If x^{-1} is defined by $xx^{-1}=1$ (so that $(x^{-1})^{-1}=x$), (1) with $z=x^{-1}$ gives $yx=(yx^{-1}\cdot x)x$, $y=yx^{-1}\cdot x$ and

$$(3) \qquad yx\cdot x^{-1} = y, \qquad R(x)^{-1} = R(x^{-1}).$$

Let $\mathfrak{G}$ be the group generated by the $R(x)$, and consider its elements $S=R(a_1)R(a_2)\cdots R(a_n)$, $T=R(a_1)^2R(a_2)^2\cdots R(a_n)^2$. By (3), every element of $\mathfrak{G}$ can be put in the form S. By repeated application of (1), $yS\cdot zS=(yz)T$. If the a_i are chosen so that $1S=1$, let $y=1$ and have $S=T$. Thus *the subgroup $\mathfrak{J}$ of $\mathfrak{G}$, consisting of the S with $1S=1$, is a group of automorphisms of G.* We use this "remark" several times; its value lies in the readily verified fact that the elements left invariant by a set of automorphisms of a loop form a subloop.

Let H be the subloop of the theorem and H_1 the subset consisting of the z in H such that $ab\cdot z=a\cdot bz$. Equivalently, $zS=z$ where $S=R(ab)R(a^{-1})R(b^{-1})$. By the remark, S induces an automorphism of H, so H_1 is a subloop of H. Moreover H_1 contains c, by hypothesis,

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[1] Numbers in brackets refer to the references cited at the end of the paper.

and a, b, by (2). Hence $H_1 = H$.

In particular, therefore, $ab \cdot c^{-1} = a \cdot bc^{-1}$. If we apply (3) and (1) in turn, $ab = (a \cdot bc^{-1})c$, $ab \cdot c = ac \cdot b$. It is now easy to see that the relation $ab \cdot c = a \cdot bc$ remains true under all permutations of a, b, c. We deduce among other things that $ac \cdot z = a \cdot cz$ for all z in H.

Let H_2 be the subset consisting of the y in H such that $ay \cdot z = a \cdot yz$ for all z in H. By the remark, H_2 is a subloop of H, containing a, by (2), and b, c, by the above proofs. Hence $H_2 = H$.

A similar argument now gives $xy \cdot z = x \cdot yz$ for all x, y, z in H.

References

1. R. H. Bruck, *Contributions to the theory of loops*, Trans. Amer. Math. Soc. vol. 60 (1946) pp. 245–354.

2. Ruth Moufang, *Zur Struktur von Alternativkörpern*, Math. Ann. vol. 110 (1935) pp. 416–430.

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