DEFERRED BERNSTEIN POLYNOMIALS

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1. **Introduction.** If $f(x)$ is continuous in $[0, 1]$, its Bernstein polynomial

$$(1) \qquad B_n^f(x) = \sum_{\nu=0}^{n} f(\nu/n) p_{\nu n}(x), \qquad p_{\nu n}(x) = C_{n,\nu} x^\nu (1-x)^{n-\nu},$$

converges to $f(x)$ uniformly in $[0, 1]$. The relation $B_n^f(x) \to f(x)$ still holds for some classes of discontinuous functions (Chlodowsky [2], Herzog and Hill [3]),[1] but no fairly general results can exist in this direction, since the values of $f(x)$ at rational points do not, in general, suffice to describe properly the behaviour of $f(x)$.

In this note we replace (1) by

$$(2) \qquad B_n^{f_a}(x) = \sum_{\nu=0}^{n} f(\nu/n - a) p_{\nu n}(x),$$

where $f(x)$ is supposed periodic with period 1 and $f_a(x) = f(x-a)$. One may expect that the values of a, for which (2) does not behave regularly, are exceptional. In §2 we consider the case where $f(x)$ has one single point c of unboundedness; we may assume that $c=0$. We show that in this case the behaviour of (2) depends on the degree of transcendency of a. We prove a gap theorem in §3. Finally in §4 we make some remarks about the relation of our problem to the convergence of deferred Riemann sums.

2. We need a lemma.

Lemma. *Let $0<x<1$ and $\delta>0$ be fixed numbers. Then there is a $0<q<1$ and an integer n_0 such that*

$$(3) \qquad \sum_{|\nu/n - x| \geq \delta, 0 \leq \nu \leq n} p_{\nu n}(x) \leq q^n, \qquad n \geq n_0.$$

Proof. Since nq^n is smaller than $(q+\epsilon)^n$ for large n, it is sufficient to prove the inequality for a single $p_{\nu n}(x)$. We may further confine our attention to ν's such that ν/n is to the right of x. We have

$$\frac{p_{\mu+1,n}(x)}{p_{\mu n}(x)} = \frac{n-\mu}{\mu+1} \frac{x}{1-x} < \left(\frac{1}{\mu/n} - 1\right) \Big/ \left(\frac{1}{x} - 1\right),$$

Received by the editors October 28, 1949.

[1] Numbers in brackets refer to the references at the end of the paper.

and thus for $\delta/2 \leqq \mu/n - x < \delta$,

$$p_{\mu+1,n}(x)/p_{\mu n}(x) \leqq \left(\frac{1}{x+\delta/2} - 1\right)\Big/\left(\frac{1}{x} - 1\right) = q_1 < 1.$$

There are at least $\delta n/2 - 2$ such μ; and if μ_0 is the smallest of them, for $\nu/n - x \geqq \delta$ we have

$$p_{\nu n}(x) \leqq p_{\mu_0 n}(x) q_1^{\delta n/2-2} \leqq q^n, \qquad 0 < q < 1.$$

Let $A(u)$ be a positive function of $u > 0$. A real number a *admits of the approximation* $A(u)$ (Koksma [5]) if there are an infinity of integers n such that $|\nu/n - a| \leqq A(n)$ is fulfilled for some integer ν. Almost all real a do not admit of $A(u)$, if $A(u) = u^{-k}$, $k > 2$.

THEOREM 1. *Let $f(x)$ be a function with period 1 which is continuous in $0 < x < 1$; let*

$$|f(x)| = O(\exp(|x|^{-1/(2+\epsilon)})), \qquad x \to 0, \tag{4}$$

for some positive $\epsilon > 0$. Then for almost all a

$$\lim_{n\to\infty} B_n^{f_a}(x) = f_a(x) = f(x - a) \tag{5}$$

holds for all $x \neq a$.

PROOF. Almost all of a do not admit of the approximation $u^{-(2+\epsilon/2)}$. For any such a and an $x \neq a$, choose $0 < \sigma < |x - a|$ and put $|x - a| - \sigma = \delta > 0$. If $\bar{f}(t)$ is defined by $\bar{f}(t) = 0$ for $|t - a| < \sigma$ and $\bar{f}(t) = f(t - a)$ for $|t - a| \geqq \sigma$, we have $B_n^{\bar{f}}(x) \to \bar{f}(x) = f(x - a)$, since $\bar{f}$ is continuous at $t = x$. Now

$$\begin{aligned} |B_n^{f_a}(x) - B_n^{\bar{f}}(x)| &\leqq \sum_{|\nu/n - a| < \sigma} |f(\nu/n - a)| \, p_{\nu n}(x) \\ &\leqq \sum_{|\nu/n - x| > \delta} |f(\nu/n - a)| \, p_{\nu n}(x). \end{aligned}$$

As $|\nu/n - a| \geqq n^{-(2+\epsilon/2)}$ for all large n, we obtain, by the lemma,

$$|B_n^{f_a}(x) - B_n^{\bar{f}}(x)| \leqq C q^n \exp n^{(2+\epsilon/2)/(2+\epsilon)} \to 0, \qquad n \to \infty,$$

with some constant C. This proves (5).

Theorem 1 is applicable for instance to the functions $f(x) = x^{-\gamma}$, $\gamma > 0$. Bernstein polynomials of the functions $(x - a)^{-\gamma}$ were also discussed by Wigert [9], who did not arrive at definitive results.

THEOREM 2. *If in addition to the hypotheses of Theorem 1, $f(x)$*

tends to infinity for $x\to 0$, there is a set of the power of the continuum of the a's such that the sequence $B_n^{f_a}(x)$ is unbounded for any $x\neq a$, 0, 1.

PROOF. There exists a function $\psi(x)$, tending monotonically to $+\infty$ for $x\to 0$, such that $|f(x)|\geqq\psi(|x|)$. Let ν_0 be the index for which $|\nu_0/n-a|<1/2n$; there is at most one such ν_0. Let $\bar{f}(t)$ be defined as in the proof of Theorem 1, and let $\bar{B}_n$ be the sum obtained from $B_n^{f_a}$ by omitting the term u corresponding to $\nu=\nu_0$. As above, we have $\bar{B}_n\to f(x-a)$ for $x\neq a$. Thus it is sufficient to prove that the term $u=f(\nu_0/n-a)p_{\nu_0 n}(x)$ is unbounded. Since the smallest of the $p_{\nu n}(x)$ is $p_{0n}=x^n$ or $p_{nn}=(1-x)^n$, we have

$$|u|\geqq\psi(|\nu_0/n-a|)p_{\nu_0 n}(x)\geqq q^n\psi(|\nu_0/n-a|),\qquad 0<q<1.$$

The set of the a's admitting of the approximation $A(u)$ is for any positive function $A(u)$ of the power of the continuum, provided $A(u)\to 0$ for $u\to\infty$. We choose $A(u)$ such that

$$\psi(A(n))q^n\to+\infty.$$

Then for an infinity of n, $|\nu_0/n-a|\leqq A(n)$, and $|u|\geqq\psi(A(n))q^n\to\infty$ for these n, which completes the proof.

3. For the sake of completeness we mention the following theorem.

THEOREM 3. *If $f\epsilon L_r$, $r\geqq 1$, then*

$$\int_0^1 |B_n^{f_a}(x)|^r da\leqq\int_0^1 |f(a)|^r da, \tag{6}$$

$$\int_0^1 |B_n^{f_a}(x)-f(x-a)|^r da\to 0\qquad \textit{uniformly in } x. \tag{7}$$

From (6) we see that, for a fixed x, $B_n^{f_a}(x)$ is a linear operator mapping L_r into itself with a norm less than or equal to 1. Since $B_n^{f_a}(x)\to f(x-a)$ uniformly in x for any continuous f, (7) follows from (6) by a well known theorem on sequences of linear operators, and (6) is easily proved by using Hölder's inequality.

From Theorem 3 we see that for any integrable $f(x)$,

$$\lim_{k\to\infty} B_{n_k}^{f_a}(x)=f(x-a) \tag{8}$$

for almost all a, x, if $n_k\to\infty$ is a properly chosen sequence depending on f. A sequence n_k independent of f is provided by the following theorem.

THEOREM 4. *Suppose that $f(x)\sim\sum_{m=-\infty}^{+\infty} c_m e^{2\pi imx}$ is the Fourier series* *of* $f(x)$ *in* $[0, 1]$ *and that*

$$\sum_{m\neq 0} c_m^2 \log |m| < +\infty. \tag{9}$$

Then (8) *holds for any sequence* n_k *such that*

$$n_{k+1}/n_k \geqq \lambda > 1. \tag{10}$$

PROOF. The Fourier series of $B_n^{f_a}(x)$ as a function of a is

$$B_n^{f_a}(x) = \sum_{\nu=0}^{n} f(\nu/n - a)p_{\nu n}(x) \sim \sum_{m=-\infty}^{+\infty} c_m e^{-2\pi i m a} \sum_{\nu=0}^{n} e^{2\pi i m\nu/n} p_{\nu n}(x),$$

and that of $B_n^{f_a}(x) - f(x-a)$ is

$$\sum_{m=-\infty}^{+\infty} c_m e^{-2\pi i m a}[B_n^{\phi_m}(x) - \phi_m(x)], \qquad \phi_m(x) = e^{2\pi i m x}.$$

Therefore, by Parseval's identity we have

$$\begin{aligned}\int_0^1 \int_0^1 | B_n^{f_a}(x) - f(x-a)|^2 da\,dx \\ = \sum_{m=-\infty}^{+\infty} c_m^2 \int_0^1 | B_n^{\phi_m}(x) - \phi_m(x)|^2 dx \\ = \sum c_m^2 \alpha_{mn}\end{aligned} \tag{11}$$

say. Since the functions $\phi_m(x)$ have a continuous second derivative, there is a $0 \leqq \xi \leqq 1$ such that (Bernstein [1])

$$B_n^{\phi_m}(x) - \phi_m(x) = \frac{x(1-x)}{n} \phi_m''(\xi)$$

and we obtain

$$| B_n^{\phi_m}(x) - \phi_m(x)| \leqq C m^2 n^{-1} \tag{12}$$

with a constant C. We have also the trivial inequality

$$| B_n^{\phi_m}(x) - \phi_m(x)| \leqq 1. \tag{13}$$

From (12) and (13) we see that

$$\sum_{k=1}^{\infty} \alpha_{mn_k} = \sum_{n_k \leqq m^2} + \sum_{n_k > m^2} \leqq \sum_{n_k \leqq m^2} 1 + C m^4 \sum_{n_k > m^2} n_k^{-2}. \tag{14}$$

The first sum has no more than $\log m^2/\log \lambda + 1 = O(\log |m|)$ terms;

and the second sum is equal to $O(m^{-4})$. The whole sum (14) is therefore equal to $O(\log |m|)$. From (11) we see that $\sum_k \int\int |B_{n_k} - f_a|^2 da dx$ converges, which completes the proof.

4. It may be worth while to remark that since $\int_0^1 p_{\nu n}(x)dx = (n+1)^{-1}$, formal integration of the relation $B_n^{f_a}(x) \to f(x-a)$ leads to

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} f(\nu/n - a) = \int_0^1 f(x)dx. \tag{15}$$

This relation was discussed by several authors (Jessen [4], Marcinkiewicz and Salem [6], Salem [8]). The conditions on f used in Theorem 4 are stronger than those under which (15) is known to hold. But for Theorem 1 the matter is just reversed, and the corresponding assertion for Riemann sums is not true even for the functions $f(x) = |x|^{-\gamma}$, $\gamma \geqq 1/2$ (Marcinkiewicz and Zygmund [7], where the case $f(x) = |x|^{-1/2} \log |x|$ is discussed).

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