BERNSTEIN POLYNOMIALS FOR FUNCTIONS OF TWO VARIABLES OF CLASS $C^{(k)}$

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Introduction. Let $\phi(x, y)$ be a continuous function of the real variables x and y, where x and y are in the closed region R: $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$. The Bernstein polynomial $B_{mn}(x, y)$ associated with the function $\phi(x, y)$ is defined as

$$(1) \qquad B_{mn}(x, y) = \sum_{p=0}^{n} \sum_{q=0}^{m} \phi\left(\frac{p}{n}, \frac{q}{m}\right) \lambda_{n,p}(x) \lambda_{m,q}(y)$$

where $\lambda_{n,p}(x) = C_{n,p} x^p (1-x)^{n-p}$, $\lambda_{m,q}(y) = C_{m,q} y^q (1-y)^{m-q}$.

A function $\phi(x, y)$ is said to be of class $C^{(k)}$ for x and y in R, if the partial derivatives of order $1, 2, \cdots, k$ of $\phi(x, y)$ exist and are continuous. We shall use the notation

$$\phi^{(i,k-i)} = \frac{\partial^{(k)} \phi(x, y)}{\partial x^i \partial y^{k-i}} \qquad (i = 0, 1, \cdots, k)$$

and, for brevity, shall omit functional arguments from expressions whenever possible.

It is the purpose of this paper to prove the

THEOREM. *If $\phi(x, y)$ is of class $C^{(k)}$ for x and y in R, then*

$$\lim_{m,n=\infty} B_{mn}^{(i,k-i)} = \phi^{(i,k-i)}$$

and the convergence is uniform in R.

This theorem, for functions of one variable and for $k = 0$, was proved by S. Bernstein [1],[2] and again for functions of one variable but for arbitrary k, the theorem was proved by S. Wigert [2]. The process of extending the results of Bernstein and Wigert to functions of two variables of class $C^{(k)}$ introduces aspects which are of interest.

Preliminary results. We shall make use of the relations

$$(2) \qquad \sum_{p=0}^{n} \lambda_{n,p}(x) = 1,$$

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

$$\sum_{p=0}^{n} (nx - p)^2 \lambda_{n,p}(x) = nx(1 - x) \qquad [3, \text{p. } 152]. \tag{3}$$

If we define, for $k \geqq 0$, $i = 0, 1, \cdots, k$,

$$A_{p,q}^{(i,k-i)} = \sum_{\alpha=0}^{i} \sum_{\beta=0}^{k-i} (-1)^{\alpha+\beta} C_{i,\alpha} C_{k-i,\beta} \phi\left(\frac{p + (i - \alpha)}{n}, \frac{q + (k - i - \beta)}{m}\right),$$

then, by mathematical induction, the following two lemmas can be established.

LEMMA 1. *If* $0 \leqq i \leqq k$, $i \leqq n$, $k \leqq m$, *x and y in R, then the kth partial derivatives of the Bernstein polynomials* (1) *are given by*

$$B_{mn}^{(i,k-i)} = \frac{n!m!}{(n - i)!(m - k + i)!} \sum_{p=0}^{n-i} \sum_{q=0}^{m-k+i} A_{p,q}^{(i,k-i)} \lambda_{n-i,p} \lambda_{m-k+i,q}.$$

LEMMA 2. *If* $0 \leqq i \leqq k$, $0 \leqq p \leqq n - i$, $0 \leqq q \leqq m - k + i$, *and if* $\phi(x, y)$ *is of class* $C^{(k)}$ *for x and y in R, then there exist two real numbers* $\xi = \xi(p)$, $\gamma = \gamma(q)$ *such that* $0 < \xi < 1$, $0 < \gamma < 1$ *and such that*

$$A_{p,q}^{(i,k-i)} = \frac{1}{n^i m^{k-i}} \phi^{(i,k-i)}\left(\frac{p + \xi i}{n}, \frac{q + \gamma(k - i)}{m}\right).$$

The next lemma is basic for the proof of the theorem.

LEMMA 3. *For fixed x and y in R and for fixed positive integers M and N, let d be an arbitrary positive number and let $a(p, q)$ be a quantity dependent upon p and q and such that*

$$| a(p, q) | \leqq \pi_1 \quad \text{for} \quad \left| x - \frac{p}{N} \right| \leqq d \quad \text{and} \quad \left| y - \frac{q}{M} \right| \leqq d, \tag{4}$$

$$| a(p, q) | \leqq \pi_2 \quad \text{for} \quad \left| x - \frac{p}{N} \right| > d \quad \text{and} \quad \left| y - \frac{q}{M} \right| > d. \tag{5}$$

Furthermore, assume that it is possible to split off from $a(p, q)$ terms $a'(p)$ independent of q or terms $b'(q)$ independent of p, that is,

$$a(p, q) = a''(p, q) + a'(p) = b''(p, q) + b'(q)$$

such that

$$| a''(p) | \leqq \pi_3 \quad \text{for} \quad \left| x - \frac{p}{N} \right| \leqq d \quad \text{and} \quad \left| y - \frac{q}{M} \right| > d, \tag{6}$$

$$| a'(p, q) | \leqq \pi_4 \quad \text{for} \quad \left| x - \frac{p}{N} \right| \leqq d, \tag{7}$$

(8) $$\left| b''(p, q) \right| \leqq \pi_5 \quad \textit{for} \quad \left| x - \frac{p}{N} \right| > d \quad \textit{and} \quad \left| y - \frac{q}{M} \right| \leqq d,$$

(9) $$\left| b'(q) \right| \leqq \pi_6 \quad \textit{for} \quad \left| y - \frac{q}{M} \right| \leqq d,$$

then

(10) $$\left| \sum_{p=0}^{N} \sum_{q=0}^{M} a(p, q) \lambda_{N,p} \lambda_{M,q} \right| \leqq \pi_1 + \pi_4 + \pi_6 + \frac{\pi_2(M + N)}{8MNd^2} + \frac{\pi_3}{4Md^2} + \frac{\pi_5}{4Nd^2}.$$

Proof. Consider the inequality

(11) $$\begin{aligned} \left| \sum_{p=0}^{N} \sum_{q=0}^{M} a(p, q) \lambda_{N,p} \lambda_{M,q} \right| &\leqq \sum_{|x-p/N| \leqq d} \sum_{|y-q/M| \leqq d} \left| a(p, q) \right| \lambda_{N,p} \lambda_{M,q} \\ &\quad + \sum_{|x-p/N| > d} \sum_{|y-q/M| > d} \left| a(p, q) \right| \lambda_{N,p} \lambda_{M,q} \\ &\quad + \sum_{|x-p/N| \leqq d} \sum_{|y-q/M| > d} \left| a''(p, q) + a'(p) \right| \lambda_{N,p} + \lambda_{M,q} \\ &\quad + \sum_{|x-p/N| > d} \sum_{|y-q/M| \leqq d} \left| b''(p, q) + b'(q) \right| \lambda_{N,p} \lambda_{M,q} \\ &= s_1 + s_2 + s_3 + s_4, \end{aligned}$$

then

$$s_1 \leqq \pi_1 \sum_{|x-p/N| \leqq d} \sum_{|y-q/M| \leqq d} \lambda_{N,p} \lambda_{M,q} \qquad \text{(by (4))}$$

$$\leqq \pi_1 \sum_{p=0}^{N} \sum_{q=0}^{M} \lambda_{N,p} \lambda_{M,q} = \pi_1 \qquad \text{(by (3))}$$

and hence $s_1 \leqq \pi_1$.

The inequalities in (5) imply the inequality

$$\frac{M^2(Mx - p)^2 + N^2(My - q)^2}{2M^2N^2d^2} > 1$$

and by (5) again

$$\left| a(p, q) \right| \leqq \pi_2 < \frac{\pi_2}{2M^2N^2d^2} \left\{ M^2(Nx - p)^2 + N^2(My - q)^2 \right\},$$

thus

$$s_2 \leqq \frac{\pi_2}{2M^2N^2d^2}\sum_{p=0}^{N}\sum_{q=0}^{M}\left\{M^2(Nx-p)^2+N^2(My-q)^2\right\}\lambda_{N,p}\lambda_{M,q}$$

$$= \frac{\pi_2}{2MNd^2}\left[Mx(1-x)+Ny(1-y)\right] \qquad \text{(by (3) and (2))}$$

and since

$$\max_{x,y\in R}\left[Mx(1-x)+Ny(1-y)\right]=\frac{M+N}{4},$$

we have

$$s_2 \leqq \frac{\pi_2(M+N)}{8MNd^2}.$$

In similar fashion, using (6), (7), (8), and (9), we obtain

$$s_3 \leqq \pi_4+\frac{\pi_3}{4Md^2}, \qquad s_4 \leqq \pi_6+\frac{\pi_5}{4Nd^2}.$$

Collecting these estimates in (11) we obtain (10).

Proof of the theorem. By Lemmas 1 and 2, we have the inequality

$$\begin{aligned}
&\left| B_{mn}^{(i,k-i)}-\phi^{(i,k-i)}\right| \\
&\quad \leqq \left|\sum_{p=0}^{n-i}\sum_{q=0}^{m-k+i}\left\{1-\frac{n!m!}{(n-i)!(m-k+i)!n^i m^{k-i}}\right\}\right. \\
&\qquad \left.\cdot\phi^{(i,k-i)}\left(\frac{p+\xi i}{n},\frac{q+\gamma(k-i)}{m}\right)\lambda_{n-i,p}\lambda_{m-k+i,q}\right| \\
&\qquad +\left|\sum_{p=0}^{n-i}\sum_{q=0}^{m-k+i}\left\{\phi^{(i,k-i)}\left(\frac{p+\xi i}{n},\frac{q+k(k-i)}{m}\right)\right.\right. \\
&\qquad \left.\left.-\phi^{(i,k-i)}(x,y)\right\}\lambda_{n-i,p}\lambda_{m-k+i,q}\right| = S_1+S_2.
\end{aligned}$$

We first estimate S_2. Let $\phi^{(i,k-i)}(x,y)=\psi(x,y)$, then since $\psi(x,y)$ is continuous for x and y in R, it is uniformly continuous and bounded on R. That is, for an arbitrary positive number ϵ, there exists a number $d(\epsilon)>0$ such that if $|x-x_1|\leqq d(\epsilon)/2$ and $|y-y_1| \leqq d(\epsilon)/2$, where x_1 and y_1 are in R, then

$$|\psi(x_1,y_1)-\psi(x,y)|\leqq \epsilon/6 \tag{12}$$

and

$$|\psi(x, y)| \leqq L^{(k)}, \qquad \text{for } x \text{ and } y \text{ in } R, \tag{13}$$

where $L^{(k)} = \max_{x,y\in R} |\psi(x, y)|$.

If p, n, i are positive integers such that $0 \leqq p \leqq n-i$, and if ξ is a real number such that $0<\xi<1$, then

$$\left|\frac{p}{n-i} - \frac{p+\xi i}{n}\right| < \frac{i}{n}, \tag{14}$$

and if m, q, k, i are positive integers such that $0 \leqq q \leqq m-(k-i)$, and γ a real number such that $0<\gamma<1$, then

$$\left|\frac{q}{m-k+i} - \frac{q+\gamma(k-i)}{m}\right| < \frac{k-i}{m}. \tag{15}$$

Now if x is fixed and p is such that $0 \leqq p \leqq n-1$ and $|x-p/(n-i)| \leqq d(\epsilon)/2$, then by (14)

$$\left|x - \frac{p+\xi i}{n}\right| \leqq \frac{d(\epsilon)}{2} + \frac{i}{n}$$

and if we choose

$$N_{1\epsilon} > 2i/d(\epsilon) + i,$$

then for $n > N_{1\epsilon}$

$$i/n < d(\epsilon)/2$$

and

$$\left|x - \frac{p+\xi i}{n}\right| \leqq d(\epsilon) \qquad \text{if } n > N_{1\epsilon}. \tag{16}$$

In the same manner, choose

$$M_{1\epsilon} > (k-i) + \frac{2(k-i)}{d(\epsilon)};$$

then for fixed y and q such that $0 \leqq q \leqq m-k+i$ and $|y-q/(m-k+i)| \leqq d(\epsilon)/2$, we obtain from (15)

$$\left|y - \frac{q+\gamma(k-i)}{m}\right| \leqq d(\epsilon) \qquad \text{if } m > M_{1\epsilon}. \tag{17}$$

If we choose $x_1 = (p+\xi i)/n$, $y_1 = (q+\gamma(k-i))/m$ in (12), then from (16), (17) we have

$$\left|\psi\left(\frac{p+\xi i}{n}, \frac{q+\gamma(k-i)}{m}\right)-\psi(x, y)\right| \leqq \frac{\epsilon}{6} \tag{18}$$

if $n>N_{1\epsilon}$, $m>M_{1\epsilon}$, $|x-(p+\xi i)/n| \leqq d(\epsilon)/2$, $|y-(q+\gamma(k-i))/m| \leqq d(\epsilon)/2$.

If we now let:

Δ_{n-i} indicate summation for all p such that $|x-p/(n-i)| \leqq d(\epsilon)/2$,

Δ'_{n-i} indicate summation for all p such that $|x-p/(n-i)| > d(\epsilon)/2$,

Δ_{m-k+i} indicate summation for all q such that $|y-q/(m-k+i)| \leqq d(\epsilon)/2$,

Δ'_{m-k+i} indicate summation for all q such that $|y-q/(m-k+i)| > d(\epsilon)/2$,

then S_2 may be written as

$$\begin{aligned}
S_2 \leqq & \sum_{\Delta_{n-i}} \sum_{\Delta_{m-k+i}} \left|\psi\left(\frac{p+\xi i}{n}, \frac{q+\gamma(k-i)}{m}\right) - \psi(x, y)\right| \lambda_{n-i,p}\lambda_{m-k+i,q} \\
& + \sum_{\Delta'_{n-i}} \sum_{\Delta'_{m-k+i}} \left|\psi\left(\frac{p+\xi i}{n}, \frac{q+\gamma(k-i)}{m}\right) - \psi(x, y)\right| \lambda_{n-i,p}\lambda_{m-k+i,q} \\
& + \sum_{\Delta_{n-i}} \sum_{\Delta'_{m-k+i}} \left|\psi\left(\frac{p+\xi i}{n}, \frac{q+\gamma(k-i)}{m}\right) - \psi\left(\frac{p+\xi i}{n}, y\right)\right| \lambda_{n-i,p}\lambda_{m-k+i,q} \\
& + \sum_{\Delta_{n-i}} \sum_{\Delta'_{m-k+i}} \left|\psi\left(\frac{p+\xi i}{n}, y\right) - \psi(x, y)\right| \lambda_{n-i,p}\lambda_{m-k+i,q} \\
& + \sum_{\Delta'_{n-i}} \sum_{\Delta_{m-k+i}} \left|\psi\left(\frac{p+\xi i}{n}, \frac{q+\gamma(k-i)}{m}\right) - \psi\left(x, \frac{q+\gamma(k-i)}{m}\right)\right| \lambda_{n-i,p}\lambda_{m-k+i,q} \\
& + \sum_{\Delta'_{n-i}} \sum_{\Delta_{m-k+i}} \left|\psi\left(x, \frac{q+\gamma(k-i)}{m}\right) - \psi(x, y)\right| \lambda_{n-i,p}\lambda_{m-k+i,q}.
\end{aligned} \tag{19}$$

If we use (13) and (18) and let

$$a(p, q) = \psi\left(\frac{p + \xi i}{n}, \frac{q + \gamma(k - i)}{m}\right) - \psi(x, y),$$

$$d = d(\epsilon)/2,$$

$$\pi_1 = \pi_4 = \pi_6 = \epsilon/6,$$

$$\pi_2 = \pi_3 = \pi_5 = 2L^{(k)},$$

then the hypotheses of Lemma 3 are satisfied and (19) becomes

$$S_2 \leqq \frac{\epsilon}{2} + \frac{3L^{(k)}}{[d(\epsilon)]^2}\left(\frac{1}{n - i} + \frac{1}{m - k + i}\right), \quad \text{for } n > N_{1\epsilon}, m > M_{1\epsilon}.$$

To estimate S_1, we note that there exist two numbers $N_{2\epsilon} > i$, $M_{2\epsilon} > k - i$ such that if $n > N_{2\epsilon}$, $m > M_{2\epsilon}$, then

$$\left| 1 - \frac{n!m!}{(n - i)!(m - k + i)!n^i m^{k-i}} \right| < \frac{\epsilon}{6L^{(k)}}.$$

Thus

$$S_1 \leqq \frac{\epsilon}{6L^{(k)}} \sum_{p=0}^{n-i} \sum_{q=0}^{m-k+i} \left| \psi\left(\frac{p + \xi i}{n}, \frac{q + \gamma(k - i)}{m}\right) \right| \lambda_{n-i,p}\lambda_{m-k+i,q}$$

$$\leqq \frac{\epsilon}{6} \sum_{p=0}^{n-i} \sum_{q=0}^{m-k+i} \lambda_{n-i,p}\lambda_{m-k+i,q} \qquad \text{by (13)}$$

and hence

$$S_1 \leqq \epsilon/6.$$

Using these estimates of S_1 and S_2 we have

$$(20) \qquad \left| B_{mn}^{(i,k-i)} - \phi^{(i,k-i)} \right| \leqq \frac{2}{3}\epsilon + \frac{3L^{(k)}}{[d(\epsilon)]^2}\left[\frac{1}{n - i} + \frac{1}{m - k + i}\right]$$

for $n > N_{1\epsilon}, N_{2\epsilon}$; $m > M_{1\epsilon}, M_{2\epsilon}$.

Let

$$(21) \qquad N_{3\epsilon} > i + \frac{18L^{(k)}}{\epsilon[d(\epsilon)]^2},$$

$$(22) \qquad M_{3\epsilon} > (k - i) + \frac{18L^{(k)}}{\epsilon[d(\epsilon)]^2},$$

and take

$$N_\epsilon = \max\,(N_{1\epsilon}, N_{2\epsilon}, N_{3\epsilon}),$$
$$M_\epsilon = \max\,(M_{1\epsilon}, M_{2\epsilon}, M_{3\epsilon}).$$

In (21) let $n > N_\epsilon$ and such that

$$\frac{3L^{(k)}}{[d(\epsilon)]^2(n-i)} < \frac{\epsilon}{6}$$

and in (22) let $m > M_\epsilon$ and such that

$$\frac{3L^{(k)}}{[d(\epsilon)]^2(m-k+i)} < \frac{\epsilon}{6}.$$

Thus, for $n > N_\epsilon$ and $m > M_\epsilon$, (20) becomes

$$\left| B_{mn}^{(i,k-i)} - \phi^{(i,k-i)} \right| < \epsilon$$

and the theorem is proved.

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