REMARKS ON A THEOREM OF E. J. MCSHANE

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In a recent paper E. J. McShane [3] has given a theorem which is the common core of a variety of results about Baire sets, Baire functions, and convex sets in topological spaces including groups and linear spaces. In general terms his theorem states that if \( \mathcal{J} \) is a family of open maps defined in one topological space \( X_1 \) into another, \( X_2 \), the total image \( \mathcal{J}(S) \) of a second category Baire set \( S \) in \( X_1 \) has, under certain conditions on \( \mathcal{J} \) and \( S \), a nonvacuous interior. The point of these remarks is to show that his argument yields a theorem for a larger class than the second category Baire sets. From this there follow slightly stronger and more specific versions of some of his results, including his principal theorem, as well as a proof that if \( S \) is a subset of a weak sort of topological group and \( S \) contains a second category Baire set, then the identity element lies in the interior of both \( S^{-1}S \) and \( SS^{-1} \). There is also at the end an extension of Zorn's theorem on the structure of certain semigroups.

In a topological space \( X \) let the closure and interior of a set \( E \) be denoted by \( E^* \) and \( E^\circ \) and the null set by \( \Lambda \). For any set \( S \) let \( I(S) = \bigcup \{ G \mid G \text{ open}, G \cap S \text{ is first category} \} \) and \( III(S) = X - I(S) \), and let \( III(S) \) be the open set \( II(S)^\circ \cap I(X - S) \). By a fundamental theorem of Banach [2], \( S \cap I(S)^* \) is first category and hence \( S \) is second category if and only if \( II(S)^\circ \neq \Lambda \). From these we note that if \( N \) is a non-null open subset of \( III(S) \), then \( N - S \) is in the first category set \( I(X - S) \) and \( N \cap S \) cannot be first category since \( N \) is non-null open and disjoint with \( I(S) \). This gives us the following lemma.

**Lemma 1.** For any non-null open subset \( N \) of \( III(S) \), the sets \( N - S \) and \( N \cap S \) are first and second category respectively.

We recall that \( S \) is defined to be a Baire set in \( X \) if \( (S - G) \cup (G - S) \) is first category for some open set \( G \); an equivalent condition is \( II(S)^\circ \subseteq I(X - S) \), or \( II(S)^\circ = III(S) \).

**Definition.** \( \mathcal{B}(X) \) is the family of all second category Baire sets in \( X \). A set \( S \) is in \( \mathcal{B}(X) \) if and only if \( S \) contains an element of \( \mathcal{B}(X) \).
Any member $S$ of $\mathfrak{A}(X)$ is characterized by $I(X-S) \supseteq II(S) \neq \emptyset$ or by $III(S) = II(S) \neq \emptyset$. A characterization of the class $\mathfrak{A}(X)$, which is generally larger (for example, when $X$ is the reals), is given by the following lemma.

**Lemma 2.** $S \in \mathfrak{A}(X)$ is equivalent to each of these: (1) $III(S) \neq \emptyset$, (2) there is a non-null open set $N$ such that $N - S$ is first category and $N \cap S$ is second category.

If $S \in \mathfrak{A}(X)$, let $B \in \mathfrak{B}(X)$ with $B \subseteq S$. The latter implies $II(S) \neq \emptyset$, $I(X-S) \supseteq I(X-B)$ and hence $III(S) = III(B) \neq \emptyset$. When (1) holds, the set $N = III(S)$ satisfies (2) by Lemma 1 above. When (2) is true, let $B = N \cap S$. Then $B$ is second category and differs from the open set $N$ only on the first category set $N - S$; thus $B \in \mathfrak{B}(X)$.

Now let $X_1$ and $X_2$ be topological spaces and $f = [f]$ a non-null family of functions defined in $X_1$ to $X_2$ where the domain of definition $D_f$ of each $f$ is open in $X_1$ and each $f$ is open on its domain to $X_2$, that is, maps open sets into open sets. For any sets $E_1 \subseteq X_1$ and $E_2 \subseteq X_2$ we shall write $\mathcal{J}(E_1)$ for the set $\bigcup \{f(E_1 \cap D_f) \mid f \in F\}$ and $\mathcal{J}^{-1}(E_2)$ for the set of all $x \in X_1$ such that $f(x)$ is defined and in $E_2$ for some $f \in F$.

For any pair of sets $S_0$ and $N$ in $X_1$ consider these conditions on $S_0$, $N$, and $\mathcal{J}$: (i) $S_0 \neq \emptyset$ and $S_0 \subseteq D_f$ for each $f \in \mathcal{J}$; (ii) $N$ is open and $N \supseteq S_0$; (iii) $f(N \cap D_f) \subseteq \mathcal{J}(S_0)$ for each $f \in \mathcal{J}$. We note that (iii) is equivalent to the following: (iii') $y \in \mathcal{J}(N)$ implies $S_0 \cap \mathcal{J}^{-1}(y) \neq \emptyset$. The essential proposition in McShane's theorem can be stated as follows.

**Lemma 3.** When $\mathcal{J}$, $S_0$, and $N$ satisfy (i), (ii), and (iii), $\mathcal{J}(S_0)$ is non-null and open.

For (following McShane) if $U = \bigcup f(N \cap D_f)$, then from the original assumptions on $\mathcal{J}$ clearly $U$ is open in $X_2$, and $U \subseteq \mathcal{J}(S_0)$ from (iii). By (i) and (ii) on the other hand, $S_0 \subseteq D_f \cap N$ for each $f$ and hence $\mathcal{J}(S_0) \subseteq U$. And $\mathcal{J}(S_0) \neq \emptyset$ since $S_0$ and $\mathcal{J}$ are nonvacuous and $S_0 \subseteq D_f$ for all $f$.

Now suppose these are true for two sets $N$ and $S$ in $X_1$: (i') $\Lambda \neq N \cap S \subseteq D_f$ for all $f$ in $\mathcal{J}$; (ii') $N$ is open. If we set $S_0 = N \cap S$ clearly (i) and (ii) hold; hence we have the following lemma.

**Lemma 4.** If $\mathcal{J}$, $S$, and $N$ satisfy (i') and (ii') and also either (iii) or (iii') with $S_0 = N \cap S$, then $\mathcal{J}(N \cap S)$ is a non-null open set.

Obviously (iii') is true for $S_0 = N \cap S$ if $N \cap S \cap \mathcal{J}^{-1}(y)$ is second
category for each \( y \in \mathcal{J}(N) \); and the latter clearly holds whenever \( N - S \) is first category and the following is true: (iii\(^{''}\)) \( y \in \mathcal{J}(N) \) implies \( N \cap \mathcal{J}^{-1}(y) \) is second category. Lemma 4 now gives us the following lemma.

**Lemma 5.** \( \mathcal{J}(N \cap S) \) is non-null and open whenever \( \mathcal{J}, S, \) and \( N \) satisfy (i\(^{'}\)), (ii\(^{'}\)), and (iii\(^{''}\)) and \( N - S \) is first category.

From Lemmas 1 and 5 we have immediately the following theorem.

**Theorem.** \( \mathcal{J}(N \cap S) \) is non-null and open provided that \( N \) is a non-null open subset of \( III(S) \), that \( D_f \supseteq S \cap N \) for each \( f \) in \( \mathcal{J} \), and that (iii\(^{''}\)) is satisfied.

In view of Lemma 2 the theorem is concerned with and only with elements \( S \) of \( \mathfrak{A}(X) \). When \( S \) is in the more restricted class \( \mathfrak{A}(X) \), that is, when \( III(S) = II(S)^0 \neq \Lambda \), the following slightly sharper version of McShane's theorem results on taking \( N \) to be \( II(S)^0 \).

**Corollary 1.** If \( S \in \mathfrak{A}(X) \), if \( D_f \supseteq S \cap II(S)^0 \) for each \( f \) in \( \mathcal{J} \), and if \( II(S)^0 \cap \mathcal{J}^{-1}(y) \) is second category for each \( y \in \mathcal{J}(N) \), then \( \mathcal{J}(S) = \mathcal{J}(S \cap II(S)^0) \cup \mathcal{J}(S \cap I(S)^*) \) where \( \mathcal{J}(S \cap II(S)^0) \) is non-null and open in \( X_2 \) and \( S \cap I(S)^* \) is first category in \( X_3 \).

Now let \( X \) be a group having a topology in which \( xy \) is continuous in each variable and let \( e \) be the identity element. We recall the following properties of the function \( II(E) \) in any topological space \([2, pp. 46–47]\): (a) \( II(E) \) is always closed, \( II(II(E)) = II(E) \), and \( II(E) \subseteq II(F) \) when \( E \subseteq F \); (b) \( II(E)^0 = II(E) \subseteq E^* \); (c) for any open set \( G \), \( II(G \cap II(E)^0) \supseteq G \cap II(E) \); and (d) for any homeomorphism \( \phi \) in \( X \), \( II(\phi(E)) = (II(E)) \).

**Corollary 2.** Let \( R \) be second category in \( X \) and \( S \in \mathfrak{A}(X) \). Suppose \( G \) and \( H \) are open and \( G \cap II(R) \neq \Lambda \neq H \cap III(S) \). If we set \( A = G \cap R \cap II(R) \) and \( B = H \cap S \cap III(S) \), it follows that \( A^{-1}B \) and \( BA^{-1} \) are non-null open subsets of \( R^{-1}S \) and \( SR^{-1} \) respectively.

For each \( a \in A \) define \( f_a(x) = a^{-1}x \) for all \( x \) in the non-null open set \( N = H \cap III(S) \) and set \( \mathcal{J} = [f_a] \). Since \( G \cap II(R) \neq \Lambda \) it follows from (\( \gamma \)) above that \( A \neq \Lambda \). Thus \( \mathcal{J} \) is non-null, each \( f_a \) is open on its open domain, and \( D_{f_a} \supseteq S \cap N = B \) for each \( f_a \). If it is shown that \( N \cap \mathcal{J}^{-1}(y) \) is second category whenever \( y \in \mathcal{J}(N) \), the theorem is applicable and \( \mathcal{J}(N \cap S) = \mathcal{J}(B) = A^{-1}B \) is a non-null open subset of \( R^{-1}S \). If \( y \in \mathcal{J}(N) \), then \( y = a^{-1}x \) for some \( a \in A \) and \( x \in N \), and \( \mathcal{J}^{-1}(y) = Aa^{-1}x \). Hence \( II(\mathcal{J}^{-1}(y)) = II(Aa^{-1}x) = II(A)Aa^{-1}x \) by (\( \delta \)) above; since \( II(A) \supset A \) by (\( \gamma \)), it follows that \( II(\mathcal{J}^{-1}(y)) \ni x \). Then, since \( N \ni x \) and is open,
$N \cap f^{-1}(y)$ must be second category. A similar proof, setting $f_a(x) = x a^{-1}$, establishes the theorem's other assertion.

Taking $G = H = X$ we have the following corollaries.

**Corollary 3.** Let $S \subseteq \mathfrak{A}(X)$. If $R$ is second category then $[R \cap II(R)^{-1}][S \cap III(S)]$ and $[S \cap III(S)][R \cap II(R)^{-1}]$ are non-null open subsets of $R^{-1}S$ and $SR^{-1}$ respectively.

**Corollary 4.** If $S \subseteq \mathfrak{A}(X)$ and $R^{-1}$ is second category the sets $RS$ and $SR$ have non-null interiors.

**Corollary 5.** When $S \subseteq \mathfrak{A}(X)$ it follows that $e \subseteq (S^{-1}S)^0 \cap (SS^{-1})^0$.

**Corollary 6.** If $S$ is a subgroup and $S \subseteq \mathfrak{A}(X)$, then $S = S^0$ and hence, since $S$ is a subgroup, $S = S^*$.

Corollaries 4 and 6 are slight extensions of results of McShane and Banach [3]. Corollary 5 for a more restricted $X$ is in another paper [4].

**Corollary 7.** Suppose $E^{-1}$ is second category in $X$ whenever $E$ is second category. If $R$ is second category and $S \subseteq \mathfrak{A}(X)$, then

1. $[R \cap II(R^{-1})][S \cap III(S)]$ and $[S \cap III(S)][R \cap II(R^{-1})^{-1}]$ are non-null open subsets of $RS$ and $SR$ respectively, and

2. $II(R)III(S)^* \subseteq ((RS)^0)^*$ and $III(S)^*II(R) \subseteq ((SR)^0)^*$.

Conclusion (1) results immediately from Corollary 3 since $R^{-1}$ is second category. To establish (2) let $C = R \cap II(R^{-1})^{-1}$ and $D = S \cap III(S)$, and note that (1) implies that $C*D^* \subseteq (CD)^* \subseteq ((RS)^0)^*$ and $D^*C^* \subseteq ((SR)^0)^*$. It is thus sufficient to show that $II(R) \subseteq C^*$ and $III(S) \subseteq D^*$. The latter follows from (γ) above; for on setting $G = I(X-S)$ and $E = S$ therein we have $II(D) \supseteq III(S)$ and hence $D^* \supseteq III(S)$. For the former, consider any open set $N$ intersecting $II(R)$. The set $N \cap R$ is second category and hence $(N \cap R)^{-1}$ is a second category subset of $R^{-1}$. From this it follows that $(N \cap R)^{-1} \cap II(R^{-1}) \neq \emptyset$, for otherwise $(N \cap R)^{-1}$ is in the first category set $R^{-1} \cap I(R^{-1})$. Taking inverses we have $\Lambda \neq N \cap R \cap II(R^{-1})^{-1} = N \cap C$, proving that $II(R) \subseteq C^*$.

When $S \subseteq \mathfrak{A}(X)$ the terms $III(S)$ and $III(S)^*$ in Corollary 7 can be replaced by $II(S)^0$ and $II(S)$, and hence in particular $II(S)^2 \subseteq ((S^0)^0)^*$. If also $S$ is a semigroup, that is, $S^2 \subseteq S$, then $II(S)^4 \subseteq (S^0)^*$. It may also be remarked that the first assumption in Corollary 7 is weaker than that of assuming $x^{-1}$ to be continuous in $x$, as is shown by the reals with intervals $a \leq x < b$ as neighborhoods.
Lemma 6. For any sets $R$ and $S$ in $X$ let $\Gamma(R, S) = \Pi(R)S^* \cup R^*II(S) \cup II(R)II(S^*) \cup II(R^*)II(S)$ and $\Delta(R, S) = II(R)S^* \cup R^*II(S) \cup II(R)II(S)$. Then $(RS)^* \supseteq II(RS) \supseteq \Gamma(R, S) \supseteq \Delta(R, S) \supseteq II(R)II(S)$.

For any $s \in S$ and $r \in R$ we have $II(R)s = II(Rs) \subseteq II(RS)$ and $rII(S) = II(rs) \subseteq II(RS)$, so that $II(R)S \cup II(R)S^* \subseteq II(RS)$. Since $II(RS)$ is closed and $A^*B^* \subseteq (AB)^*$ for any $A$ and $B$, it follows that $II(R)S^* \cup R^*II(S) \subseteq II(RS)$. Moreover $II(RS) = II(RS) \supseteq II(R)II(S^*) \cup II(R^*)II(S)$. But from what has already been shown we have $II(R)II(S^*) \supseteq II(R)II(S^*) \cup II(R)II(S^*) = II(R)S^* \cup R^*II(S)$, and similarly $II(R^*)II(S) \supseteq II(R^*)II(S) \cup R^*II(S)$. Thus $II(RS) \supseteq \Gamma(R, S)$. The rest is obvious.

From this it is clear that $II(S) \supseteq \Gamma(S, S) \supseteq \Delta(S, S) \supseteq II(S)^2$ for any $S$. Another consequence is the following corollary.

Corollary 8. Suppose $E^{-1}$ is second category whenever $E$ is second category. If $R$ is second category, $S \in \mathcal{B}(X)$, and $S \supseteq R \cup S$, then $(S^o)^* \supseteq \Gamma(R, S) \cup \Gamma(S, R) \cup \Gamma(R, S) \cup \Gamma(S, R)]II(R)$.

Clearly $(S^o)^* \supseteq (RS)^o$, and from a remark after Corollary 7, $(RS)^o \supseteq II(R)II(S)$. At the same time, obviously $II(S) \supseteq II(RS) \cup II(SR)$, where $II(RS) \supseteq \Gamma(R, S)$ and $II(SR) \supseteq \Gamma(S, R)$ by Lemma 6. Thus $(S^o)^* \supseteq II(R) \cup \Gamma(S, R) \cup \Gamma(R, S) \cup \Gamma(S, R)]II(R)$. Similarly, $(S^o)^* \supseteq (RS)^o \supseteq \Gamma(R, S) \cup \Gamma(S, R) \cup \Gamma(R, S) \cup \Gamma(S, R)]II(R)$.

When $S_1$ is a semigroup, this has obvious consequences, first when $S_1 \in \mathcal{B}(X)$ and $R = S = S_1$ and second when $S_1$ is in $\mathcal{B}(X)$ and $R = S_1^{-1}$ and $S = X - S_1$. These together imply Corollary 5 of [3].

Lemma 7. For any subset $S$ of a topological space $X$ these conditions are equivalent: (1) $S$ is a Baire set, $S \subseteq II(S)$, and $X - S \subseteq II(X - S)$; (2) the equalities (i) $S^o = II(S)^o = (S^*)^o = I(X - S)$ and (ii) $S^* = II(S) = (S^o)^o = I(X - S)^*$ are true; (3) equalities (i) and (ii) hold when $S$ and $X - S$ are interchanged.

Taking complements in (i) and (ii) yields (ii) and (i) with $S$ and $X - S$ interchanged; thus (2) and (3) are equivalent. Concerning (1) and (2) we note that by (2) above $S \subseteq II(S)$ is equivalent to (4) $S^* = II(S)$, that $X - S \subseteq II(X - S)$ is equivalent to $(X - S)^* = II(X - S)$, that is, to (5) $S^o = I(X - S)$, and recall that $S$ is a Baire set if and only if (6) $II(S)^o \subseteq I(X - S)$. Obviously (2) now implies (1). Conversely, (1) implies (4), (5), and (6), where (6) implies (7) $II(S) \subseteq I(X - S)^*$ since $II(S) = (II(S)^o)^*$. From (4), (7), and (5) we have $S^* = II(S) \subseteq I(X - S)^* = (S^o)^*$; since $(S^o)^* \subseteq S^*$, (ii) follows.
Taking interiors in (ii) yields \((S^*)^0 = II(S)^0 = (I(X-S))^0\); since 
\((I(X-S))^0 = ((X-II(X-S))^0) = (X-II(X-S))^0 = X-II(X-S) = I(X-S)\) and (5) holds, (i) now follows also.

Lemmas 6 and 7, which are independent of the other lemmas and corollaries, provide the following mild extension of Zorn's theorem on the structure of a semigroup \(S\) when \(S\) is a Baire set such that \(S\) and \(S^{-1}\) are second category at \(e\) [1, pp. 157–158; 3, Corollary 6].

**Theorem.** Suppose \(S\) is a Baire set and \(S \supseteq RS\) or \(S \supseteq SR\) for some \(R\) such that \(II(R)^m \cap II(R^{-1})^n \supseteq e\) for some \(m\) and \(n \geq 1\). Then (2) and (3) of Lemma 7 are true.

Suppose \(S \supseteq RS\). From Lemma 6 and the definition of \(\Delta(R, S)\), \(S^* \supseteq II(S) \supseteq II(RS) \supseteq \Delta(R, S) \supseteq II(R)S^*\), so that \(S^* \supseteq II(S) \supseteq II(R)S^*\). Multiplying by \(II(R)^k\) we have \(II(R)^kS^* \supseteq II(R)^{k+1}S^*\), and hence \(S^* \supseteq II(S) \supseteq II(R)^kS^*\) for any \(k \geq 1\). When \(e \in II(R)^m\), it is then clear that \(S^* \supseteq II(S) \supseteq S^*\), or \(S^* = II(S)\). Since \(S \supseteq RS\) implies \(X-S \supseteq R^{-1}(X-S)\), we also have \((X-S)^* = II(X-S)\) in case \(e \in II(R^{-1})^n\) for some \(n \geq 1\). A similar proof applies when \(S \supseteq SR\). Thus our hypotheses here imply (1) of Lemma 7, and the present conclusion follows.

**References**


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