REMARKS ON A THEOREM OF E. J. McSHANE[1]

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In a recent paper E. J. McShane [3][2] has given a theorem which is the common core of a variety of results about Baire sets, Baire functions, and convex sets in topological spaces including groups and linear spaces. In general terms his theorem states that if $\mathcal{J}$ is a family of open maps defined in one topological space X_1 into another, X_2, the total image $\mathcal{J}(S)$ of a second category Baire set S in X_1 has, under certain conditions on $\mathcal{J}$ and S, a nonvacuous interior. The point of these remarks is to show that his argument yields a theorem for a larger class than the second category Baire sets. From this there follow slightly stronger and more specific versions of some of his results, including his principal theorem, as well as a proof that if S is a subset of a weak sort of topological group and S contains a second category Baire set, then the identity element lies in the interior of both $S^{-1}S$ and SS^{-1}. There is also at the end an extension of Zorn's theorem on the structure of certain semigroups.

In a topological space X let the closure and interior of a set E be denoted by E^* and E^0 and the null set by Λ. For any set S let $I(S) = \bigcup[G \mid G$ open, $G \cap S$ is first category$]$ and $II(S) = X - I(S)$, and let $III(S)$ be the open set $II(S)^0 \cap I(X-S)$. By a fundamental theorem of Banach [2], $S \cap I(S)^*$ is first category and hence S is second category if and only if $II(S)^0 \neq \Lambda$. From these we note that if N is a non-null open subset of $III(S)$, then $N-S$ is in the first category set $(X-S) \cap I(X-S)$, and $N \cap S$ cannot be first category since N is non-null open and disjoint with $I(S)$. This gives us the following lemma.

LEMMA 1. *For any non-null open subset N of $III(S)$, the sets $N-S$ and $N \cap S$ are first and second category respectively.*

We recall that S is defined to be a Baire set in X if $(S-G) \cup (G-S)$ is first category for some open set G; an equivalent condition is $II(S)^0 \subset I(X-S)$, or $II(S)^0 = III(S)$.

DEFINITION. *$\mathfrak{B}(X)$ is the family of all second category Baire sets in X. A set S is in $\mathfrak{A}(X)$ if and only if S contains an element of $\mathfrak{B}(X)$.*

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[2] Numbers in brackets refer to the references at the end of the paper.

Any member S of $\mathfrak{B}(X)$ is characterized by $I(X-S)\supset II(S)^0\neq\Lambda$ or by $III(S)=II(S)^0\neq\Lambda$. A characterization of the class $\mathfrak{A}(X)$, which is generally larger (for example, when X is the reals), is given by the following lemma.

LEMMA 2. *$S\in\mathfrak{A}(X)$ is equivalent to each of these*: (1) $III(S)\neq\Lambda$, (2) *there is a non-null open set N such that $N-S$ is first category and $N\cap S$ is second category.*

If $S\in\mathfrak{A}(X)$, let $B\in\mathfrak{B}(X)$ with $B\subset S$. The latter implies $II(S)^0 \supset II(B)^0$ and $I(X-S)\supset I(X-B)$ and hence $III(S)\supset III(B)\neq\Lambda$. When (1) holds, the set $N=III(S)$ satisfies (2) by Lemma 1 above. When (2) is true, let $B=N\cap S$. Then B is second category and differs from the open set N only on the first category set $N-S$; thus $B\in\mathfrak{B}(X)$.

Now let X_1 and X_2 be topological spaces and $\mathcal{J}=[f]$ a non-null family of functions defined in X_1 to X_2 where the domain of definition D_f of each f is open in X_1 and each f is open on its domain to X_2, that is, maps open sets into open sets. For any sets $E_1\subset X_1$ and $E_2\subset X_2$ we shall write $\mathcal{J}(E_1)$ for the set $\bigcup[f(E_1\cap D_f)\mid f\in F]$ and $\mathcal{J}^{-1}(E_2)$ for the set of all $x\in X_1$ such that $f(x)$ is defined and in E_2 for some $f\in\mathcal{J}$.

For any pair of sets S_0 and N in X_1 consider these conditions on S_0, N, and $\mathcal{J}$: (i) $S_0\neq\Lambda$ and $S_0\subset D_f$ for each $f\in\mathcal{J}$; (ii) N is open and $N\supset S_0$; (iii) $f(N\cap D_f)\subset\mathcal{J}(S_0)$ for each $f\in\mathcal{J}$. We note that (iii) is equivalent to the following: (iii′) $y\in\mathcal{J}(N)$ implies $S_0\cap\mathcal{J}^{-1}(y)\neq\Lambda$. The essential proposition in McShane's theorem can be stated as follows.

LEMMA 3. *When $\mathcal{J}$, S_0, and N satisfy* (i), (ii), *and* (iii), *$\mathcal{J}(S_0)$ is non-null and open.*

For (following McShane) if $U=\bigcup_f f(N\cap D_f)$, then from the original assumptions on $\mathcal{J}$ clearly U is open in X_2, and $U\subset\mathcal{J}(S_0)$ from (iii). By (i) and (ii) on the other hand, $S_0\subset D_f\cap N$ for each f and hence $\mathcal{J}(S_0)\subset U$. And $\mathcal{J}(S_0)\neq\Lambda$ since S_0 and $\mathcal{J}$ are nonvacuous and $S_0\subset D_f$ for all f.

Now suppose these are true for two sets N and S in X_1: (i′) $\Lambda\neq N\cap S\subset D_f$ for all f in $\mathcal{J}$; (ii′) N is open. If we set $S_0=N\cap S$ clearly (i) and (ii) hold; hence we have the following lemma.

LEMMA 4. *If $\mathcal{J}$, S, and N satisfy* (i′) *and* (ii′) *and also either* (iii) *or* (iii′) *with $S_0=N\cap S$, then $\mathcal{J}(N\cap S)$ is a non-null open set.*

Obviously (iii′) is true for $S_0=N\cap S$ if $N\cap S\cap\mathcal{J}^{-1}(y)$ is second category for each $y \in \mathcal{J}(N)$; and the latter clearly holds whenever $N-S$ is first category and the following is true: (iii'') $y \in \mathcal{J}(N)$ implies $N \cap \mathcal{J}^{-1}(y)$ is second category. Lemma 4 now gives us the following lemma.

LEMMA 5. *$\mathcal{J}(N \cap S)$ is non-null and open whenever $\mathcal{J}$, S, and N satisfy* (i'), (ii'), *and* (iii'') *and $N-S$ is first category.*

From Lemmas 1 and 5 we have immediately the following theorem.

THEOREM. *$\mathcal{J}(N \cap S)$ is non-null and open provided that N is a non-null open subset of $III(S)$, that $D_f \supset S \cap N$ for each f in $\mathcal{J}$, and that* (iii'') *is satisfied.*

In view of Lemma 2 the theorem is concerned with and only with elements S of $\mathfrak{A}(X_1)$. When S is in the more restricted class $\mathfrak{B}(X_1)$, that is, when $III(S) = II(S)^0 \neq \Lambda$, the following slightly sharper version of McShane's theorem results on taking N to be $II(S)^0$.

COROLLARY 1. *If $S \in \mathfrak{B}(X_1)$, if $D_f \supset S \cap II(S)^0$ for each f in $\mathcal{J}$, and if $II(S)^0 \cap \mathcal{J}^{-1}(y)$ is second category for each $y \in \mathcal{J}(N)$, then $\mathcal{J}(S) = \mathcal{J}(S \cap II(S)^0) \cup \mathcal{J}(S \cap I(S)^*)$ where $\mathcal{J}(S \cap II(S)^0)$ is non-null and open in X_2 and $S \cap I(S)^*$ is first category in X_1.*

Now let X be a group having a topology in which xy is continuous in each variable and let e be the identity element. We recall the following properties of the function $II(E)$ in any topological space [2, pp. 46–47]: (α) $II(E)$ is always closed, $II(II(E)) = II(E)$, and $II(E) \subset II(F)$ when $E \subset F$; (β) $(II(E)^0)^* = II(E) \subset E^*$; ($\gamma$) for any open set G, $II(G \cap E \cap II(E)^0) \supset G \cap II(E)$; and ($\delta$) for any homeomorphism ϕ in X, $II(\phi(E)) = \phi(II(E))$.

COROLLARY 2. *Let R be second category in X and $S \in \mathfrak{A}(X)$. Suppose G and H are open and $G \cap II(R) \neq \Lambda \neq H \cap III(S)$. If we set $A = G \cap R \cap II(R)$ and $B = H \cap S \cap III(S)$, it follows that $A^{-1}B$ and BA^{-1} are non-null open subsets of $R^{-1}S$ and SR^{-1} respectively.*

For each $a \in A$ define $f_a(x) = a^{-1}x$ for all x in the non-null open set $N = H \cap III(S)$ and set $\mathcal{J} = [f_a]$. Since $G \cap II(R) \neq \Lambda$ it follows from (γ) above that $A \neq \Lambda$. Thus $\mathcal{J}$ is non-null, each f_a is open on its open domain, and $D_{f_a} \supset S \cap N = B$ for each f_a. If it is shown that $N \cap \mathcal{J}^{-1}(y)$ is second category whenever $y \in \mathcal{J}(N)$, the theorem is applicable and $\mathcal{J}(N \cap S) = \mathcal{J}(B) = A^{-1}B$ is a non-null open subset of $R^{-1}S$. If $y \in \mathcal{J}(N)$, then $y = a^{-1}x$ for some $a \in A$ and $x \in N$, and $\mathcal{J}^{-1}(y) = Aa^{-1}x$. Hence $II(\mathcal{J}^{-1}(y)) = II(Aa^{-1}x) = II(A)a^{-1}x$ by (δ) above; since $II(A) \supset A$ by (γ), it follows that $II(\mathcal{J}^{-1}(y)) \ni x$. Then, since $N \ni x$ and is open,

$N\cap\mathcal{J}^{-1}(y)$ must be second category. A similar proof, setting $f_a(x)=xa^{-1}$, establishes the theorem's other assertion.

Taking $G=H=X$ we have the following corollaries.

COROLLARY 3. *Let $S\in\mathfrak{A}(X)$. If R is second category then $[R\cap II(R)]^{-1}[S\cap III(S)]$ and $[S\cap III(S)][R\cap II(R)]^{-1}$ are non-null open subsets of $R^{-1}S$ and SR^{-1} respectively.*

COROLLARY 4. *If $S\in\mathfrak{A}(X)$ and R^{-1} is second category the sets RS and SR have non-null interiors.*

COROLLARY 5. *When $S\in\mathfrak{A}(X)$ it follows that $e\in(S^{-1}S)^0\cap(SS^{-1})^0$.*

COROLLARY 6. *If S is a subgroup and $S\in\mathfrak{A}(X)$, then $S=S^0$ and hence, since S is a subgroup, $S=S^*$.*

Corollaries 4 and 6 are slight extensions of results of McShane and Banach [3]. Corollary 5 for a more restricted X is in another paper [4].

COROLLARY 7. *Suppose E^{-1} is second category in X whenever E is second category. If R is second category and $S\in\mathfrak{A}(X)$, then*

(1) $[R\cap II(R^{-1})^{-1}][S\cap III(S)]$ *and* $[S\cap III(S)][R\cap II(R^{-1})^{-1}]$

are non-null open subsets of RS and SR respectively, and

(2) $II(R)III(S)^*\subset((RS)^0)^*$ *and* $III(S)^*II(R)\subset((SR)^0)^*$.

Conclusion (1) results immediately from Corollary 3 since R^{-1} is second category. To establish (2) let $C=R\cap II(R^{-1})^{-1}$ and $D=S\cap III(S)$, and note that (1) implies that $C^*D^*\subset(CD)^*\subset((RS)^0)^*$ and $D^*C^*\subset((SR)^0)^*$. It is thus sufficient to show that $II(R)\subset C^*$ and $III(S)\subset D^*$. The latter follows from (γ) above; for on setting $G=I(X-S)$ and $E=S$ therein we have $II(D)\supset III(S)$ and hence $D^*\supset III(S)$. For the former, consider any open set N intersecting $II(R)$. The set $N\cap R$ is second category and hence $(N\cap R)^{-1}$ is a second category subset of R^{-1}. From this it follows that $(N\cap R)^{-1}\cap II(R^{-1})\neq\Lambda$, for otherwise $(N\cap R)^{-1}$ is in the first category set $R^{-1}\cap I(R^{-1})$. Taking inverses we have $\Lambda\neq N\cap R\cap II(R^{-1})^{-1}=N\cap C$, proving that $II(R)\subset C^*$.

When $S\in\mathfrak{B}(X)$ the terms $III(S)$ and $III(S)^*$ in Corollary 7 can be replaced by $II(S)^0$ and $II(S)$, and hence in particular $II(S)^2\subset((S^2)^0)^*$. If also S is a semigroup, that is, $S^2\subset S$, then $II(S)^2\subset(S^0)^*$. It may also be remarked that the first assumption in Corollary 7 is weaker than that of assuming x^{-1} to be continuous in x, as is shown by the reals with intervals $a\leq x<b$ as neighborhoods.

LEMMA 6. *For any sets R and S in X let* $\Gamma(R, S) = II(R)S^* \cup R^*II(S) \cup II(R)II(S^*) \cup II(R^*)II(S)$ *and* $\Delta(R, S) = II(R)S^* \cup R^*II(S) \cup II(R)II(S)$. *Then* $(RS)^* \supset II(RS) \supset \Gamma(R, S) \supset \Delta(R, S) \supset II(R)II(S)$.

For any $s \in S$ and $r \in R$ we have $II(R)s = II(Rs) \subset II(RS)$ and $rII(S) = II(rS) \subset II(RS)$, so that $II(R)S \cup RII(S) \subset II(RS)$. Since $II(RS)$ is closed and $A^*B^* \subset (AB)^*$ for any A and B, it follows that $II(R)S^* \cup R^*II(S) \subset II(RS)$. Moreover $II(II(E)) = II(E)$ for any E; hence $II(RS) = II(II(RS)) \supset II(II(R)S^*) \cup II(R^*II(S))$. But from what has already been shown we have $II(II(R)S^*) \supset II(II(R))(S^*)^* \cup II(R)^*II(S^*) = II(R)S^* \cup II(R)II(S^*)$, and similarly $II(R^*II(S)) \supset II(R^*)II(S) \cup R^*II(S)$. Thus $II(RS) \supset \Gamma(R, S)$. The rest is obvious.

From this it is clear that $II(S^2) \supset \Gamma(S, S) \supset \Delta(S, S) \supset II(S)^2$ for any S. Another consequence is the following corollary.

COROLLARY 8. *Suppose* E^{-1} *is second category whenever E is second category. If R is second category,* $S \in \mathfrak{B}(X)$, *and* $S \supset RS \cup SR$, *then*

$$(S^0)^* \supset II(R)[\Gamma(R, S) \cup \Gamma(S, R)] \cup [\Gamma(R, S) \cup \Gamma(S, R)]II(R).$$

Clearly $(S^0)^* \supset ((RS)^0)^*$, and from a remark after Corollary 7, $((RS)^0)^* \supset II(R)II(S)$. At the same time, obviously $II(S) \supset II(RS) \cup II(SR)$, where $II(RS) \supset \Gamma(R, S)$ and $II(SR) \supset \Gamma(S, R)$ by Lemma 6. Thus $(S^0)^* \supset II(R)[\Gamma(R, S) \cup \Gamma(S, R)]$. Similarly, $(S^0)^* \supset ((SR)^0)^* \supset [\Gamma(S, R) \cup \Gamma(R, S)]II(R)$.

When S_1 is a semigroup, this has obvious consequences, first when $S_1 \in \mathfrak{B}(X)$ and $R = S = S_1$ and second when S_1 is in $\mathfrak{B}(X)$ and $R = S_1^{-1}$ and $S = X - S_1$. These together imply Corollary 5 of [**3**].

LEMMA 7. *For any subset S of a topological space X these conditions are equivalent*: (1) *S is a Baire set,* $S \subset II(S)$, *and* $X - S \subset II(X - S)$; (2) *the equalities* (i) $S^0 = II(S)^0 = (S^*)^0 = I(X - S)$ *and* (ii) $S^* = II(S) = (S^0)^* = I(X - S)^*$ *are true*; (3) *equalities* (i) *and* (ii) *hold when S and* $X - S$ *are interchanged.*

Taking complements in (i) and (ii) yields (ii) and (i) with S and $X - S$ interchanged; thus (2) and (3) are equivalent. Concerning (1) and (2) we note that by (β) above $S \subset II(S)$ is equivalent to (4) $S^* = II(S)$, that $X - S \subset II(X - S)$ is equivalent to $(X - S)^* = II(X - S)$, that is, to (5) $S^0 = I(X - S)$, and recall that S is a Baire set if and only if (6) $II(S)^0 \subset I(X - S)$. Obviously (2) now implies (1). Conversely, (1) implies (4), (5), and (6), where (6) implies (7) $II(S) \subset I(X - S)^*$ since $II(S) = (II(S)^0)^*$. From (4), (7), and (5) we have $S^* = II(S) \subset I(X - S)^* = (S^0)^*$; since $(S^0)^* \subset S^*$, (ii) follows.

Taking interiors in (ii) yields $(S^*)^0 = II(S)^0 = (I(X-S)^*)^0$; since $(I(X-S)^*)^0 = ((X-II(X-S))^*)^0 = (X-II(X-S)^0)^0 = X-(II(X-S)^0)^* = X-II(X-S) = I(X-S)$ and (5) holds, (i) now follows also.

Lemmas 6 and 7, which are independent of the other lemmas and corollaries, provide the following mild extension of Zorn's theorem on the structure of a semigroup S when S is a Baire set such that S and S^{-1} are second category at e [**1**, pp. 157–158; **3**, Corollary 6].

THEOREM. *Suppose S is a Baire set and $S \supset RS$ or $S \supset SR$ for some R such that $II(R)^m \cap II(R^{-1})^n \ni e$ for some m and $n \geqq 1$. Then* (2) *and* (3) *of Lemma* 7 *are true.*

Suppose $S \supset RS$. From Lemma 6 and the definition of $\Delta(R, S)$, $S^* \supset II(S) \supset II(RS) \supset \Delta(R, S) \supset II(R)S^*$, so that $S^* \supset II(S) \supset II(R)S^*$. Multiplying by $II(R)^k$ we have $II(R)^k S^* \supset II(R)^{k+1} S^*$, and hence $S^* \supset II(S) \supset II(R)^k S^*$ for any $k \geqq 1$. When $e \in II(R)^m$, it is then clear that $S^* \supset II(S) \supset S^*$, or $S^* = II(S)$. Since $S \supset RS$ implies $X-S \supset R^{-1}(X-S)$, we also have $(X-S)^* = II(X-S)$ in case $e \in II(R^{-1})^n$ for some $n \geqq 1$. A similar proof applies when $S \supset SR$. Thus our hypotheses here imply (1) of Lemma 7, and the present conclusion follows.

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