NON-MEASURABLE SETS


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NON-MEASURABLE SETS AND THE EQUATION

\[ f(x + y) = f(x) + f(y) \]

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1. A set of \( S \) real numbers which has inner measure \( m_*(S) \) different from its outer measure \( m^*(S) \) is non-measurable. An extreme form, which we shall call saturated non-measurability, occurs when \( m_*(S) = 0 \) but \( m^*(SM) = m(M) \) for every measurable set \( M \), \( m(M) \) denoting the measure of \( M \). This is equivalent to: both \( S \) and its complement have zero inner measure.

More generally, if a fixed set \( B \) of positive measure is under consideration, a subset \( S \) of \( B \) will be called \( s \)-non-mble. if both \( S \) and its complement relative to \( B \) have zero inner measure. This implies \( m_*(S) = 0 \), \( m^*(S) = m(B) \) but is implied by these conditions only if \( m(B) \) is finite.

Our object, in part, is to show that if \( B \) is either the set of all real numbers or any half-open finite interval, then for every infinite cardinal \( k \leq C \) (the power of the continuum), \( B \) can be partitioned into \( k \) disjoint subsets which are \( s \)-non-mble. and are mutually congruent under translation (modulo the length of \( B \) in the case that \( B \) is a finite interval). Sierpinski and Lusin¹ have partitioned \( B \) into continuum many disjoint \( s \)-non-mble. subsets but they are not constructed to be congruent under translation. Other well known constructions do partition \( B \) into a countable infinity of mutually congruent non-measurable subsets, but the subsets are not constructed to be saturated non-measurable.²

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² See Hahn and Rosenthal, *Set functions*, University of New Mexico Press, 1948, pp. 102–104. The construction of §8.3.3 on p. 102 (as will be shown below) does give an \( s \)-non-mble. set but this is not proved there.
2. Various authors have shown that if the axiom of choice is assumed, there are discontinuous solutions of the equation \( f(x+y) = f(x) + f(y) \) and that these solutions all have certain pathological properties. They are unbounded and non-measurable on every interval, indeed for all numbers \( a, b \) the sets \( E[x: f(x) < a] \) and \( E[x: f(x) > b] \) have zero inner measure, which implies that they are s-non-mble. A deeper theorem of Ostrowski\(^3\) shows that if \( a < b \), then the set \( E[x: f(x) < a \text{ or } f(x) > b] \) has zero inner measure, which implies that the set \( E[x: a \leq f(x) < b] \) is s-non-mble.

Now suppose that \( f(x) \) is such a discontinuous solution and that \( d \) is any number greater than 0 for which \( f(x_0) = d \) for some \( x_0 \). Let \( S_n \) denote the set \( E[x: nd \leq f(x) < d(n+1)] \). Then the \( S_n \), \( n = 0, \pm 1, \pm 2, \cdots \), are a partition of the set of all real numbers into a countable infinity of disjoint s-non-mble. sets which are congruent under translation since \( S_{n+1} = x_0 + S_n \). If \( B \) is the interval \( a \leq x < b \) and \( f(x) \) has the property \( f(b-a) = 0 \), then the \( BS_n \) are a partition of this kind of \( B \) (modulo the length of \( B \)).

3. The theorem of Ostrowski does not state that the set \( E(c) = E[x: f(x) = c] \) is s-non-mble. and this is not true in general. However, we have the following theorem.

**Theorem.** If \( f(x) \) is a discontinuous solution of \( f(x+y) = f(x) + f(y) \) which assumes only a countable number of distinct values, \( c_1, c_2, \cdots, c_n, \cdots \), and \( E_n \) denotes \( E(c_n) \), then the \( E_n \) are a partition of the set of all real numbers into disjoint, s-non-mble. sets which are congruent under translation. If \( B \) is an interval \( a \leq x < b \) and \( f(x) \) has the additional property \( f(b-a) = 0 \), then the \( BE_n \) are such a partition of \( B \) (modulo the length of \( B \)).

**Proof.** If \( x_n \) is an \( x \) for which \( f(x) = c_n \), then \( E_m = (x_m - x_n) + E_n \) so that the \( E_n \) are congruent under translation. They are obviously disjoint, and their set-union is the set of all real numbers. It remains to show that they are s-non-mble.

If \( f(x) \neq 0 \) for all \( x \neq 0 \), then \( E(0) \) would consist of one number, namely 0; each \( E_n \) would consist of one number since it is a translation of \( E(0) \), which would give the false result that the set of all real numbers is countable. This shows that \( f(\delta) = 0 \) for some \( \delta > 0 \) and hence for all \( r\delta \), \( r \) rational but arbitrary.

Let \( E_n(\alpha, \beta) \) denote the set of \( x \) in \( E_n \) for which \( \alpha \leq x < \beta \). Now \( E_n(0, \delta) \) is not a null set, for if it were so we could deduce that

$E_n(m\delta, (m+1)\delta)$ (by translation) then $E_n$ by set-union, and finally the set of all real numbers (by set-union of all $E_n$) were null sets. Hence $m^*(E_n(0, \delta)) = h\delta$ for some $0 < h \leq 1$ and some fixed $n$. It follows that $m^*E_n(r\delta, s\delta) = h(s-r)\delta$ for all rational $r$, $s$ with $r < s$; this is clear for $r = 0$ and $s$ equal to a positive integer, or to the reciprocal of a positive integer, or, finally, to any positive rational; by translation the statement then follows for all rational $r$, $s$. Now, from the continuity of $m^*$ it follows that $m^*(E_n(0, \delta)) = m(I)$ for all intervals $I$.

Suppose, if possible, that $h < 1$. Then the complement of $E_n$ has positive inner measure and hence there is an interval $I$ such that $m^*(I - E_n I) > (1 - h)m(I)$. This gives $m^*(E_n I) < hm(I)$, a contradiction. Thus $h = 1$, and $m^*(E_n I) = m(I)$ for all intervals $I$. This implies that $E_n$ is $s$-non-mble.

4. Let $e_1, e_2, \ldots, e_a, \ldots$ be any Hamel's basis so that each $x$ has a unique expression $x = \sum_a x_a e_a$ where the $x_a$ are all rational and at most a finite number of them differ from zero. Let $S_r$ be the set of $x$ for which $x_1$ equals a given rational $r$. Then if $f(x)$ is defined to be $x_1$ for every $x$, the values of $f(x)$ will be countable and §3 above shows that the $S_r$ are all $s$-non-mble. and a partition (into a countable infinity of subsets) of the desired kind of the set of all real numbers (for a finite interval $B$ it suffices to choose $e_2$ equal to the length of $B$). This does not require the theorem of Ostrowski or any other theorem on the pathology of discontinuous solutions except as proved in §3 above.

5. If $e_1, e_2, \ldots, e_a, \ldots$ is any Hamel's basis, then any set of conditions on the $x_a$ involving at most a countable number of $a$ will give a set of $x$ which is either empty, or the set of all real numbers, or $s$-non-mble. For let $f(x)$ be defined to be a solution of $f(x+y) = f(x) + f(y)$ with $f(e_a) = e_a$ if the value of $x_a$ is involved in the given conditions, and 0 otherwise. Then each set of precise restrictions on the $x_a$ involved in the original conditions, $x_a = r_a$, for arbitrary rationals $r_a$, defines an $s$-non-mble set. The conclusion follows.

In particular, the conditions $x_a = 0$ for $n = 1, 2, \ldots$ gives an $s$-non-mble set as do the conditions $x_n > 0$ for $n = 1, 2, \ldots$. It follows that if the set of all real numbers be considered as a group $G$ with the rational numbers as multipliers (operators), then a proper subgroup $G_1$ which admits the rationals as operators is $s$-non-mble. if the number of cosets of $G_1$ with respect to $G$ is countable.

\footnote{In this connection, see Jacobsthal and Knopp, Sitzungsberichte der Berliner Mathematischen Gesellschaft vol. 14 (1915) p. 121.}
6. Let \( \Omega \) be the smallest ordinal number with continuum many predecessors and let the collection of all perfect sets be arranged as a sequence \( P_\alpha, 1 \leq \alpha < \Omega. \) For each \( 1 \leq \alpha < \Omega \) define \( a_\alpha, b_\alpha \) by induction so that \( a_\alpha, b_\alpha \) are elements of \( P_\alpha \) and the \( a_\beta, b_\beta \) with \( \beta \leq \alpha \) are linearly independent with respect to rational numbers as coefficients; if an interval \( B: a \leq x < b \) is under consideration, define \( a_\alpha = b - a. \) Such \( a_\alpha, b_\alpha \) exists since for each \( \alpha, P_\alpha \) has continuum elements whereas there are less than continuum rational-linear combinations of the \( a_\beta, b_\beta \) with \( \beta < \alpha. \) Then there exists a Hamel's basis containing all \( a_\alpha \) and all \( b_\alpha \) as members. If \( k \) is any infinite cardinal less than or equal to \( C, \) there is a subset \( H_1 \) of some of the \( b_\alpha \) with cardinal \( k. \) Let \( S \) consist of all linear combinations of elements of \( H - H_1, \) using rational numbers as coefficients, and let \( y \) be an arbitrary linear combination of elements of \( H_1 \) with rational coefficients. The sets \( S + y \) form a decomposition of the whole line into \( k \) disjoint, congruent (under translation; modulo length of \( B \) if an interval \( B \) is under consideration) subsets. Since each \( S + y \) has points in every perfect set \( (S \) contains all \( a_\alpha), \) it follows that the \( S + y \) are saturated non-mble.

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