ON CONFORMAL MAPPING\(^*\) OF REGIONS BOUNDED
BY SMOOTH CURVES

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1. Introduction. The object of this note is the proof of the following theorem.

THEOREM. Suppose \( C \) is a simple closed curve which contains the origin of its interior \( R \) and which satisfies the following hypotheses:
(a) \( C \) possesses a continuously turning tangent and the tangent angle \( \alpha(s) \) considered as a function of the arc length \( s \) has the modulus of continuity \( \beta(t) \), that is,

\[
| \alpha(s + t) - \alpha(s) | \leq \beta(t), \quad t > 0,
\]

where \( \beta(t) \) is a nondecreasing function of \( t \) and \( \lim_{t \to 0} \beta(t) = 0 \).
(b) There exists a constant \( k \) such that if \( P_1 \) and \( P_2 \) are two points of \( C \) and \( \Delta s \) is the (shorter) arc between them, then

\[
\frac{\Delta s}{P_1 P_2} \leq k.
\]

(c) The diameter of \( C \) does not exceed \( D \), and the distance of the origin from \( C \) is at least equal to \( \sigma, \sigma > 0 \).

Suppose that \( w = f(z) \) maps the circle \( |z| < 1 \) conformally onto \( R \) such that \( f(0) = 0 \). Then, for every \( p > 0 \), there exists a constant \( A_p \) which depends only on \( p \), the constants \( k, D, \sigma \), and the function \( \beta(t) \)—and in no other way on the curve \( C \)—such that uniformly for \( 0 \leq p < 1 \)

\[
\left\{ \frac{1}{2\pi} \int_0^{2\pi} | f'(\rho e^{i\theta}) |^{\frac{1}{p}} d\theta \right\}^{1/p} \leq A_p.
\]

An explicit expression in terms of these parameters is obtained for \( A_p \).

The fact that the integral in (3) remains bounded for \( 0 \leq p < 1 \) under the assumption that \( C \) has continuously turning tangents was proved in an earlier paper of the writer [4, p. 362].\(^1\) The emphasis in the present note is upon the fact that the constant \( A_p \) depends only on the parameters indicated and is expressed explicitly in terms of these quantities. This result is of use when an estimate for the integral in (3) is desired which holds uniformly for the mapping functions of a family of curves \( C \).

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\(^1\) Numbers in the brackets refer to the bibliography at the end of the paper.
Before proving the theorem we state an application. By a theorem of F. Riesz, $f'(pe^{i\theta})$ has limit values as $p \to 1$ for almost all $\theta$ and (3) holds for $p=1$. By use of Hölder's inequality we have for $p > 1$, $z = e^{i\theta}$, $z_0 = e^{i\theta_0}$:

$$|f(z) - f(z_0)| = \left| \int_{\theta_0}^{\theta} f'(e^{i\theta}) ie^{i\theta} d\theta \right| \leq \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| f'(e^{i\xi}) \right|^p d\xi \right\}^{1/p} \left| \theta - \theta_0 \right| \left( \frac{\theta - \theta_0}{2\pi} \right)^{1/p} \leq A_p(2\pi)^{1/p} \left| \theta - \theta_0 \right|^{1-1/p}.$$  

From this it is easily seen that for $|z| = |z_0| = 1$

(4) $$|f(z) - f(z_0)| \leq 2^{2/p-1} A_p \pi |z - z_0|^{1-1/p},$$

and by a theorem in [6, p. 669], this inequality holds also for $|z| < 1$.

Let $\phi(w)$ denote the inverse function of $f(z)$. Then if $w_0, w$ are points of $C$ and $\Delta s$ is the length of the (shorter) arc between $w_0$ and $w$, then (note that $\phi(w)$ is an absolutely continuous function along $C$)

$$|\phi(w) - \phi(w_0)| = \left| \int_{w_0}^{w} \phi'(u) du \right| \leq \left\{ \int_{C} \left| \phi'(w) \right|^p |dw| \right\}^{1/p} (\Delta s)^{1-1/p} \leq \left\{ \int_0^{2\pi} \left| f'(e^{i\theta}) \right|^{-p} \left| f'(e^{i\theta}) \right| d\theta \right\}^{1/p} (\Delta s)^{1-1/p}. $$

Hence by (2),

(5) $$|\phi(w) - \phi(w_0)| \leq (2\pi)^{1/p} (A_{p-1})^{(p-1)/p}(k \left| w - w_0 \right|)^{1-1/p},$$

and by the theorem just quoted, this is true for all $w$ in $R$ ($w_0$ on $C$).

Combining (4) and (5) we obtain the following corollary.

**Corollary.** Under the hypotheses of the theorem there exists for every $\delta$, $0 < \delta < 1$, a constant $B$ which depends only on $\delta$ and on $k$, $D$, $\sigma$, and the function $\beta(t)$ such that for $|z_0| = 1$, $|z| \leq 1$,

$$\frac{1}{B} |z - z_0|^{1/(1-\delta)} \leq \left| f(z) - f(z_0) \right| \leq B |z - z_0|^{1-\delta}.$$

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2 This theorem is as follows: Let $R$ be a region bounded by a simple closed curve $\Gamma$ and let $f(z)$ be regular in $R$ and continuous in $R + \Gamma$. If for a point $z_0 \in \Gamma$, and for all $z \in \Gamma$: $|f(z) - f(z_0)| \leq M|z - z_0|^\alpha$, where $\alpha > 0$, then this holds also for all $z$ in $R$. 

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(If we set \( \delta = 1/p \), \( B = \text{Max} \left( 2^{2/p-1} \pi A_p, (2\pi)^{(p-1)/p} k A_{p-1} \right) \).)

The existence of a relation of this form at every point \( z_0 \) and even uniformly along the circle \( |z| = 1 \) for a fixed curve \( C \) is known [5], [1], and was proved in a different way. The point of our corollary lies again in the fact that the dependence of the constant \( B \) upon the parameters which characterize \( C \) is given.

2. **Lemmas.** We shall need the following lemmas.

**Lemma 1.** Under the hypotheses of the theorem there exists a positive \( \rho < \sigma \) which depends only on \( k, \sigma, \) and the function \( \beta(t) \), such that any circle of radius less than or equal to \( \rho \) about any point \( P \) of \( C \) intersects \( C \) in exactly two points and that the length of the subarc of \( C \) contained in this circle does not exceed \( 3\rho \).

**Proof.** Let \( \eta, 0 < \eta < k\sigma \), be so chosen that \( \beta(t) \leq 1/8 \) for \( 0 < t \leq \eta \). Then we may take \( \rho = \eta/k \). To see this we describe a circle \( K \) of radius \( r^p \rho \) about \( P \). Since \( r^p \rho < \sigma \) there are points of \( C \) exterior to \( K \), so that \( K \) intersects \( C \). Let \( P_1 \) be the first point of intersection of \( K \) and \( C \), which is met when \( C \) is traversed from \( P \) in one direction, and let \( P_2 \) be the first such point which one meets in going from \( P \) along \( C \) in the opposite direction. \( (P_1 \) and \( P_2 \) are distinct, for otherwise the curve \( C \) could not have any points in the exterior of \( K \).)

Suppose now there existed a third point of intersection of \( K \) and \( C \), say \( P_3 \). Assume, without loss of generality, that of the two complementary arcs \( PP_1P_3 \) and \( PP_2P_3 \) of \( C \), the length \( \Delta s \) of the first does not exceed that of the second. Since by (2)

\[
\Delta s \leq k \cdot \overline{PP_3} = kr \leq k\rho = \eta,
\]

it follows that for all points \( s \) of the arc \( PP_1P_3 \) (\( s_0 \) corresponds to \( P \))

\[
|\alpha(s) - \alpha(s_0)| \leq \beta(\eta) \leq 1/8.
\]

Hence, the arc \( PP_1P_3 \) of \( C \) lies within the angle which is formed by two straight lines through \( P \) each of which forms an angle of opening \( 1/8 \) with the tangent to \( C \) at \( P \). Then it follows that one of the angles between the tangent to \( C \) at \( P \) and the chord \( P_1P_3 \) of \( C \) is between \( \pi/2 - 1/8 \) and \( \pi/2 \). There exists a point \( s = s^* \) of the arc \( PP_1P_3 \) between \( P_1 \) and \( P_3 \) such that \( \alpha(s^*) \) is equal to the angle of inclination of \( P_1P_3 \). Hence \(|\alpha(s^*) - \alpha(s_0)| \geq \pi/2 - 1/8 \), which contradicts the inequality stated above.

We prove now the statement concerning the length of the subarc \( c \) of \( C \) which is contained in the interior of \( K \). Let \( z(s) = x(s) + iy(s) \) denote the parametric representation of \( C \) in terms of \( s \). Suppose that
the interval \( s_1 \leq s \leq s_2 \) corresponds to \( c \); \( z_1 = z(s_1) \), \( z_2 = z(s_2) \) are the end points of \( c \). Then

\[
z_2 - z_1 = \int_{s_1}^{s_2} (x'(s) + iy'(s))ds = (s_2 - s_1)(x'(_1) + iy'(\xi_2))
\]

where \( s_1 \leq \xi_1, \xi_2 \leq s_2 \). Hence

\[
|z_2 - z_1| = |s_2 - s_1||x'_1 + iy'\xi_1 + i(y'\xi_2 - y'\xi_1)| \\
\geq (s_2 - s_1)\{ |x'_1 + iy'\xi_1| - |y'\xi_2 - y'\xi_1| \}.
\]

Since \( x'(s) = \cos \alpha(s) \), \( y'(s) = \sin \alpha(s) \) and \( |\xi_1 - s_0| \leq \eta, |\xi_2 - s_0| \leq \eta, \)

\[
|s_2 - s_1| \leq (s_2 - s_1)\{ 1 - |y'(\xi_2) - y'(\xi_1)|\} \\
\geq (s_2 - s_1)(1 - 2\beta(\eta)) \geq 3(s_2 - s_1)/4,
\]

or

\[
s_2 - s_1 \leq \frac{4}{3} |s_2 - s_1| \leq \frac{4}{3} 2\rho < 3\rho.
\]

**Lemma 2 (Modulus of Continuity).** Suppose that the hypotheses of the theorem are satisfied. Let \( r_0 = \exp \left[ -\pi^2 D^2/2\rho^2 \right] \), where \( \rho \) is the number given in Lemma 1. Then for any two points \( z, z_0 \) of \( |z| = 1 \) for which \( |z - z_0| \leq r \leq r_0 \):

\[
|f(z) - f(z_0)| \leq \frac{\pi D}{(2 \log (1/r))^{1/2}}.
\]

**Proof.** Let \( z_0 \) be a point of \( |z| = 1 \) and \( h_r \) the part of the circle \( |z - z_0| = r \) which is contained in \( |z| < 1 \). Then by a theorem of J. Wolff [7, p. 217], there exists for every \( r, 0 < r < 1 \), an \( r_1, r < r_1 < r_1^{1/2} \), such that the image of \( h_r \) by means of \( w = f(z) \) is a cross-cut \( \gamma_r \) of \( R \) whose length \( l_1 \leq (2\pi A/\log (1/r))^{1/2} \), where \( A \) is the area of \( R \). Since \( A \leq \pi D^2/4 \), we have

(6) \[
l_1 \leq \frac{\pi D}{(2 \log (1/r))^{1/2}}.
\]

Assume now \( r \leq r_0 \). Then the region \( \{|z - z_0| < r_1, \ |z| < 1\} \) is mapped onto a subregion of \( R \) which is bounded by \( \gamma_r \) and an arc of \( C \). If \( P_1 \) and \( P_2 \) are the end points of \( \gamma_r \), then

\[
\overline{P_1P_2} \leq \frac{\pi D}{(2 \log (1/r))^{1/2}} \leq \rho < \sigma.
\]

Hence, by Lemma 1, one of the two arcs of \( C \) between \( P_1 \) and \( P_2 \), say
Because of (6), this circle contains \( \gamma_{r_1} \) and hence also the region bounded by \( \gamma_{r_1} \) and \( c_{r_1} \). Since \( \rho < \sigma \), the origin is in the exterior of this circle, and it follows that \( c_{r_1} \) is the image of the arc \( \{ |z| = 1, |z - z_0| \leq r \} \). Thus, if \( z = e^{i\theta} \) and \( |z - z_0| \leq r \), then

\[
| f(z) - f(z_0) | \leq \frac{\pi D}{(2 \log (1/r))^{1/2}} .
\]

**Lemma 3 (Properties of an auxiliary function).** Suppose

\[ 0 = \theta_1 < \theta_2 < \cdots < \theta_n < \theta_{n+1} = 2\pi \] and \( \tau_1, \tau_2, \cdots, \tau_n, \tau_{n+1} = \tau_1 + 2\pi \) are two sets of real numbers. Let \( \beta_m = (\tau_m - \tau_{m+1})/\pi, m = 1, 2, \cdots, n, \) and for \( |z| < 1 \)

\[
g(z) = \prod_{m=1}^{n} (1 - e^{-i\theta_m z})^{\beta_m};
\]

each factor is single-valued and analytic for \( |z| < 1 \) if that branch of the power function is chosen which reduces to 1 for \( z = 0 \). If this branch of \( \arg g(z) \) is taken, then

\[
\lim_{z \to e^{i\theta}} \arg g(z) = \arg g(e^{i\theta}) = \tau_m - \theta - \pi/2 - \omega, \quad \text{when} \quad \theta_m < \theta < \theta_{m+1},
\]

where

\[
\omega = \frac{1}{2\pi} \sum_{m=1}^{n} \tau_m (\theta_{m+1} - \theta_m) - \frac{3}{2}\pi.
\]

Furthermore \( \arg g(z) \) is bounded in \( |z| < 1 \).

**Proof.** We define the function \( \tau(\theta) \) by the relations \( \tau(\theta) = \tau_m \) for \( \theta_m \leq \theta < \theta_{m+1}, \) \( m = 1, 2, \cdots, n, \) \( \tau(\theta + 2\pi) = \tau(\theta) + 2\pi. \) Then we may write\(^3\) for \( |z| < 1 \):

\[
\log g(z) = \frac{1}{\pi} \sum_{m=1}^{n} (\tau_m - \tau_{m+1}) \log (1 - e^{-i\theta_m z})
\]

\[
= - \frac{1}{\pi} \int_{\theta=0}^{2\pi} \log (1 - e^{-i\theta z}) d\tau(\theta)
\]

\[
= - \frac{1}{\pi} \int_{\theta=0}^{2\pi} \log (1 - e^{-i\theta z}) [\tau(\theta) - \theta]
\]

\[
- \frac{1}{\pi} \int_{0}^{2\pi} \log (1 - e^{-i\theta z}) d\theta.
\]

\(^3\) Cf., for this representation of \( \log g(z) \) in form of a Stieltjes integral, E. Study [3].
Now \( \int_0^{2\pi} \log (1 - e^{-i\theta}z) \, d\theta = 0 \). Upon integration by parts we find

\[
\log g(z) = -\frac{1}{\pi} \left[ \log (1 - e^{-i\theta}z)(\tau(\theta) - \theta) \right]_0^{2\pi} + \frac{i\pi}{\pi} \int_0^{2\pi} \frac{\tau(\theta) - \theta}{e^{i\theta} - z} \, d\theta
\]

and since \( \tau(\theta) - \theta \) has the period \( 2\pi \), the integrated part vanishes. Since

\[
\frac{z}{e^{i\theta} - z} = \frac{1}{2} - \frac{1}{2} \frac{e^{i\theta} + z}{e^{i\theta} - z} - \frac{1}{2},
\]

we finally obtain

\[
(7) \quad \log g(z) = \frac{i}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} (\tau(\theta) - \theta) \, d\theta - \int_0^{2\pi} \frac{\tau(\theta) - \theta}{e^{i\theta} - z} \, d\theta.
\]

The last integral

\[
\int_0^{2\pi} (\tau(\theta) - \theta) \, d\theta = \frac{1}{2\pi} \sum_{m=1}^{n} \int_{\theta_m}^{\theta_{m+1}} \tau(\theta) \, d\theta - \pi
\]

\[
= \frac{1}{2\pi} \sum_{m=1}^{n} \tau_m (\theta_{m+1} - \theta_m) - \pi.
\]

The conclusion of the lemma follows easily from the representation (7) if we take the imaginary parts of both sides.

3. Proof of the theorem. (i) The correspondence between \( C \) and the unit circle \( |z| = 1 \) is given by \( w = f(e^{i\theta}), \) \( 0 \leq \theta \leq 2\pi \). Let \( s(\theta) \) denote the variable arc length of \( C \) expressed as a function of \( \theta \). By Lemma 2, the subarc \( c \) of \( C \) given by \( w = f(e^{i\theta}), \) \( \theta_1 \leq \theta \leq \theta_2, \) where \( \theta_2 - \theta_1 \leq r \leq r_0, \) lies within the circle of radius \( (\pi D/(2 \log (1/r))^{1/2} < \rho \) about the point \( w_0 = f(e^{i\theta_1}) \). By Lemma 1 the length of \( c \) is \( \leq 3\rho < 3\pi \). Since the total length of \( C \) is at least \( 2\pi \rho, \) it follows that \( c \) is the shorter arc of \( C \) between its end points, and thus by (2)

\[
0 \leq s(\theta_2) - s(\theta_1) \leq k \left| f(e^{i\theta_2}) - f(e^{i\theta_1}) \right| \leq \frac{k\pi D}{(2 \log (1/r))^{1/2}}.
\]

Let \( \tau(\theta) = a[s(\theta)] \). Given any \( \epsilon > 0 \) there exists a positive \( \delta_1 \leq k\rho \) which depends only on the function \( \beta(t) \) such that

\[
(8) \quad \beta(t) \leq \epsilon \quad \text{for} \quad 0 < t \leq \delta_1.
\]

Let \( \delta \) be so chosen that

\[
(9) \quad \frac{k\pi D}{(2 \log (1/r))^{1/2}} \leq \delta_1 \quad \text{for} \quad r \leq \delta.
\]
Then for any \( \theta_0, \theta, 0 \leq \theta, \theta_0 \leq 2\pi \), we have by (8), (9), and (1)
\[
| \tau(\theta) - \tau(\theta_0) | \leq \epsilon \quad \text{if} \quad | \theta - \theta_0 | \leq \delta.
\]

Clearly \( \delta \) depends only on \( \epsilon \) and \( D, \sigma, k, \) and the function \( \beta(t) \), as \( \delta = \exp \left\{ -k^22^\delta \right\} \) where \( \beta(\delta) \leq \epsilon, \delta_1 \leq \kappa_p. \)

(ii) Let \( n = [2\pi/\delta] + 1 \), so that \( 2\pi/n < \delta \). Let \( \theta_k = (k - 1)2\pi/n, \)
\( k = 1, 2, \ldots, n + 1 \), and \( \tau_k = \tau(\theta_k). \) We have then \( \tau_{k+1} = \tau_1 + 2\pi. \) With these two sets of numbers \( \theta_k, \tau_k \) we form the function \( g(z) \) of Lemma 3. Consider the quotient \( F(z) = (f'(z)/g(z))e^{-i\omega}, \) \( |z| < 1, \) where \( \omega \) is the constant in Lemma 3. The \( \log F(z) = \log f'(z) - \log g(z) - i\omega \) is single-valued and analytic for \( |z| < 1 \), if the same branch of \( \log g(z) \) is chosen as in Lemma 3 and if \( \log f'(z) \) is so determined that \( \log f'(0) \) is real. By a Theorem of Lindelöf [2], \( \arg f'(z), \) which is harmonic for \( |z| < 1 \), is continuous in \( |z| \leq 1 \) and
\[
\arg f'(e^{i\theta}) = \tau(\theta) - \theta - \pi/2.
\]

Hence, in every interval \( \theta_k < \theta < \theta_{k+1}, \) by (10)
\[
| \arg f'(e^{i\theta}) - \arg \left[ g(z)e^{i\omega} \right] | = | \tau(\theta) - \tau_k | \leq \epsilon.
\]

Furthermore, since \( \arg F(z) \) is bounded in \( |z| < 1 \), we have
\[
| \arg f'(z) - \arg \left[ g(z)e^{i\omega} \right] | \leq \epsilon \quad \text{for} \quad |z| < 1.
\]

(iii) Given any \( \rho > 0 \), choose \( \epsilon = 1/3\kappa \rho \) where \( \kappa = 2e/\log 2 \). Applying an inequality on conjugate functions [4, p. 356], we obtain from (11):

For \( 0 \leq \rho < 1, \)
\[
\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f'(z)e^{-i\omega}}{f'(0)g(z)e^{i\omega}} \right|^{\frac{2\rho}{1 - 2\rho}} d\theta \leq \frac{e^{2\rho \epsilon}}{1 - 2\rho \kappa} \leq \frac{e^{\epsilon/3}}{1 - 2/3} = 3e^{\epsilon/3}.
\]

Furthermore,
\[
\int_0^{2\pi} g(z)e^{i\omega} |^{\frac{2\rho}{1 - 2\rho}} d\theta = \int_0^{2\pi} \prod_{m=1}^{n} | 1 - pe^{i(\theta - \theta_m)} |^{\frac{4}{1 - pe^{i(\theta - \theta_m)}}} d\theta
\]
\[
\leq \int_0^{2\pi} \prod_{m=1}^{n} \left| 1 - pe^{i(\theta - \theta_m)} \right|^{\frac{2\rho}{1 - 2\rho}} d\theta
\]
\[
\leq \int_0^{2\pi} \prod_{m=1}^{n} 2 \csc \frac{\theta - \theta_m}{2} |^{\frac{2\rho}{1 - 2\rho}} d\theta.
\]

4 The theorem referred to is the following: Suppose \( \phi(z) = U(z) + iV(z) \) is regular for \( |z| < 1 \), suppose \( U(0) = 0 \) and \( |V(z)| = \eta \) for \( |z| < 1 \). Then there exists an absolute constant \( \kappa \), which may be taken as \( \kappa = 2e/\log 2 \), such that for every \( q, 0 < q < 1/\kappa \eta, \)
\( (1/2\pi)\int_0^{2\pi} e^{i\omega q \eta} d\theta \leq \eta^{q/(1 - q\kappa)}, 0 \leq \rho < 1. \) We apply this theorem with \( \phi(z) = \log F(z) - \log f'(0), \eta = \epsilon, q = 2\rho. \)
Since \( 2p|\beta_n| \leq 2pe/\pi < 1/\kappa \pi \), we have

\[
\frac{1}{2\pi} \int_0^{2\pi} |g(\rho e^{i\theta})|^\pm 2p d\theta
\]

\[
\leq 2^{n/\pi} \frac{1}{2\pi} \int_0^{2\pi} \prod_{m=1}^n \left| \csc \frac{\theta - \theta_m}{2} \right|^{1/\pi} d\theta = M_n,
\]

where \( M_n \) depends only on \( n \), which in turn depends only on \( p, D, \sigma, k \), and the function \( \beta(t) \).

Thus, if \( z = \rho e^{i\theta} \), we have

\[
\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f'(z)}{f'(0)} \right|^{\pm p} d\theta
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f'(z)}{f'(0)g(z)} \right|^{\pm p} g(z)^{\pm p} d\theta
\]

\[
\leq \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f'(z)}{f'(0)g(z)} \right|^{\pm 2p} d\theta \right\}^{1/2} \{ \frac{1}{2\pi} \int_0^{2\pi} |g(z)|^{\pm 2p} d\theta \}^{1/2}
\]

\[
\leq (3e^{\pi/2}M_n)^{1/2}
\]

by (12) and (13). Since by Cauchy's inequality \( \sigma \leq |f'(0)| \leq D \), we have

\[
|f'(0)|^{\pm p} \leq \text{Max } \left[ D^p, \frac{1}{\sigma^p} \right],
\]

and the conclusion of the theorem follows from (14) and (15) with

\[
A_p = [3e^{\pi/2}M_n]^{1/2p} \cdot \text{Max } [D, 1/\sigma].
\]

**Bibliography**


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