A THEOREM ON SUBHARMONIC FUNCTIONS

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A real function u is said to be subharmonic in a finite plane domain D if [**2**, p. 333][1]

(a) u is upper semi-continuous in D;

(b) $u(P) < +\infty$ for P in D;

(c) $u(P)$ is finite in a dense set in D;

(d) for every domain G with boundary B, such that $G+B \subset D$, the inequality $u(P) \leqq h(P)$ on B implies $u(P) \leqq h(P)$ in G for every function h which is harmonic in G and continuous on $G+B$.

We use the notation [**3**]

$$\Delta_r f(M) = \frac{1}{2\pi}\int_0^{2\pi} f(r, \theta)d\theta - f(M), \tag{1}$$

where (r, θ) is a polar coordinate system with pole at M, provided that $f(r, \theta) \in L$ on $[0, 2\pi]$. If $\Delta_r f(M)$ is defined for all sufficiently small r we put

$$\Lambda^* f(M) = \limsup_{r\to 0} 4\Delta_r f(M)/r^2. \tag{2}$$

$\Lambda f(M)$ and $\Lambda_* f(M)$ are defined likewise, with lim and lim inf in place of lim sup.

(3) THEOREM. *Suppose*

(i) *the function u is upper semi-continuous in the finite plane domain D;*

(ii) $\Lambda^* u(P) \geqq 0$ *for P in $D-E$, where E is at most countable;*

(iii) $\limsup_{r\to 0} \Delta_r u(P)/r \geqq 0$ *for P in E.*

Then u is subharmonic in D.

The special case of the theorem in which E is vacuous is known [**1**, p. 14]. At the end of the present paper, (3) is used to extend the results of [**3**] and [**4**].

PROOF OF (3). The existence of $\Lambda^* u(P)$ implies that $u \in L$ on every sufficiently small circle about P. Hence condition (c) is satisfied. By (ii) and (iii), condition (b) evidently also holds.

It suffices to prove (3) if (ii) is replaced by

$$\Lambda^* u(P) > 0 \qquad (P \text{ in } D - E). \tag{ii$'$}$$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

For, having established the theorem for this case, we put

$$u_m(P) = u_m(x, y) = u(x, y) + x^2/m \qquad (m = 1, 2, 3, \cdots ; P \text{ in } D),$$

so that $\Lambda^* u_m(P) = \Lambda^* u(P) + 2/m > 0$ ($m = 1, 2, 3, \cdots$; P in $D - E$), and u_m satisfies (iii). Letting $m \to \infty$, we see that u is the limit of a monotonically decreasing sequence of subharmonic functions, and is therefore itself subharmonic [2, p. 335].

We assume (ii′) to hold. For P in D, we put

$$\mu(u; P) = \underset{0 \leqq \theta < 2\pi}{\text{l.u.b.}} \limsup_{r \to 0} \frac{u(r, \theta) - u(P)}{r} \tag{4}$$

where l.u.b. stands for least upper bound, and (r, θ) is a polar coordinate system with pole at P. If the theorem is false, there exists a domain G with boundary B, $G + B \subset D$, and a function h, continuous on $G + B$, harmonic in G, and such that

$$u(P) \leqq h(P) \qquad (P \text{ on } B), \tag{5}$$

$$u(Q) > h(Q) \qquad \text{for some } Q \text{ in } G. \tag{6}$$

We may assume that G is bounded. (In fact, we can assume that G is circular [2, p. 334], but this would not simplify the proof.) Furthermore, we can choose h such that

$$|\nabla h(P)| \neq \mu(u; P) \qquad (P \text{ in } G \cdot E), \tag{7}$$

where ∇h is the gradient of h. For suppose (7) is false. Since G is bounded, we can choose a coordinate system (x, y) such that $x > 0$ for all points of G. Clearly there exists $\delta > 0$ such that (5) and (6) hold if $h(P)$ is replaced by $h_1(P) = h(P) + \epsilon x$ (P in $G + B$; $0 \leqq \epsilon \leqq \delta$). Since E is at most countable, we can choose ϵ in $[0, \delta]$ such that (7) is also satisfied by h_1.

Put $w(P) = u(P) - h(P)$. By (5) and (6), the upper semi-continuous function w attains its maximum at a point M in G. It follows that

$$\Lambda^* u(M) = \Lambda^*(w(M) + h(M)) = \Lambda^* w(M) \leqq 0,$$

which contradicts (ii′) if $M \in D - E$.

If $M \in E$, $\limsup_{r \to 0} \Delta_r w(M) r \geqq 0$. Since $w(r, \theta) \leqq w(M)$ for sufficiently small r, an application of Fatou's theorem gives

$$0 \leqq \int_0^{2\pi} \liminf_{r \to 0} \frac{w(M) - w(r, \theta)}{r} d\theta$$

$$\leqq \liminf_{r \to 0} \int_0^{2\pi} \frac{w(M) - w(r, \theta)}{r} d\theta \leqq 0.$$

It follows that

$$\limsup_{r\to 0} \frac{w(r,\theta) - w(M)}{r} = 0 \qquad \text{for almost all } \theta,$$

$$\limsup_{r\to 0} \frac{w(r,\theta) - w(M)}{r} \leqq 0 \qquad (0 \leqq \theta < 2\pi).$$

Therefore, putting $\Theta = i \cos \theta + j \sin \theta$, where i, j are unit vectors in the x, y directions, we have

$$\limsup_{r\to 0} \frac{u(r,\theta) - u(M)}{r} = \Theta \cdot \nabla h(M) \quad \text{for almost all } \theta,$$

$$\limsup_{r\to 0} \frac{u(r,\theta) - u(M)}{r} \leqq \Theta \cdot \nabla h(M) \quad (0 \leqq \theta < 2\pi).$$

Hence $\mu(u; M) = |\nabla h(M)|$, which contradicts (7). This completes the proof.

The theorem just proved enables us to derive the conclusion of Theorem I of [3] from weaker hypotheses. We obtain the following theorem.

(8) THEOREM. *Let F be continuous in the finite plane domain D. Let Z be a closed and bounded set of capacity zero. Suppose*

(i′) $\Lambda^* F(P) > -\infty$ *on* $D - D\cdot Z - E_1$, *where E_1 is at most countable, and* $\limsup_{r\to 0} \Delta_r F(P)/r \geqq 0$ *on* E_1;

(i″) $\Lambda_* F(P) < +\infty$ *on* $D - D\cdot Z - E_2$, *where E_2 is at most countable, and* $\liminf_{r\to 0} \Delta_r F(P)/r \leqq 0$ *on* E_2;

(ii) *there exists a function y, defined in D, such that $y \in L$ on every closed subset of D, and such that $y(P) \leqq \Lambda^* F(P)$ for P in D.*
Then, for all bounded domains R whose closure is contained in D, and at almost all points P of R,

$$F(P) = -\frac{1}{2\pi} \iint_R \Lambda F(Q) g(P, Q) dQ + H(P),$$

where $g(P, Q)$ is Green's function for R, and H is harmonic in R and assumes the values of F on the boundary of R.

In [3], this theorem is proved for the case $E_1 = E_2 = 0$. To prove (8), we use the notation

$$\Omega f(P) = -\frac{1}{2\pi} \iint_R f(Q) g(P, Q) dQ,$$

as in [3], and put

$$W(P) = F(P) - \Omega u(P) \qquad (P \in R),$$

where u is an upper semi-continuous function associated with y in the sense of the Vitali-Carathéodory theorem. As in (4.2.2) of [3], we have now

$$\Lambda^* W(P) \geqq 0 \quad (P \in R - R\cdot Z - R\cdot E_1). \tag{9}$$

Since u is bounded above in R, we have, by (3.5.1) of [3],

$$\begin{aligned} \limsup_{r\to 0} \frac{1}{r}\Delta_r \Omega u(P) &= \limsup_{r\to 0} \frac{1}{2\pi r}\int\int_{PQ<r} u(Q)\log\frac{r}{PQ}\,dQ \\ &\leqq \limsup_{r\to 0}\frac{mr}{4} = 0 \qquad (P\in R), \end{aligned} \tag{10}$$

where m is an upper bound of u in R. Hence

$$\begin{aligned} &\limsup_{r\to 0}\frac{1}{r}\Delta_r W(P) \\ &\geqq \limsup_{r\to 0}\frac{1}{r}\Delta_r F(P) - \limsup_{r\to 0}\frac{1}{r}\Delta_r\Omega u(P) \geqq 0 \end{aligned} \tag{11}$$

for P on $R\cdot E_1$. By (9), (11), and (3), W is subharmonic in the domain $R - R\cdot Z$. The proof then proceeds as in 4.2 of [3].

The hypotheses of Theorem II of [3] and of Theorems 2.6, 3.9, and 4.2 of [4] may be weakened in the same manner.

Finally, we wish to state the following analogue of (3).

(12) THEOREM. *Suppose*

(i) *the function f is upper semi-continuous in the segment $I = (a, b)$*;

(ii) $\limsup_{h\to 0} \{f(x+h)+f(x-h)-2f(x)\}/h^2 \geqq 0$ *for x in $I - E$, where E is at most countable*;

(iii) $\limsup_{h\to 0} \{f(x+h)+f(x-h)-2f(x)\}/h \geqq 0$ *for x in E.*

Then f is convex in (a, b).

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