A THEOREM ON SUBHARMONIC FUNCTIONS

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A real function \(u\) is said to be subharmonic in a finite plane domain \(D\) if \([2, p. 333]\)

(a) \(u\) is upper semi-continuous in \(D\);
(b) \(u(P) < +\infty\) for \(P\) in \(D\);
(c) \(u(P)\) is finite in a dense set in \(D\);
(d) for every domain \(G\) with boundary \(B\), such that \(G + B \subset D\), the inequality \(u(P) \leq h(P)\) on \(B\) implies \(u(P) \leq h(P)\) in \(G\) for every function \(h\) which is harmonic in \(G\) and continuous on \(G + B\).

We use the notation \([3]\)

\[\Delta_f(M) = -\frac{1}{2\pi} \int_0^{2\pi} f(r, \theta) d\theta - f(M),\]

where \((r, \theta)\) is a polar coordinate system with pole at \(M\), provided that \(f(r, \theta) \in L\) on \([0, 2\pi]\). If \(\Delta_f(M)\) is defined for all sufficiently small \(r\) we put

\[\Lambda^s_f(M) = \lim \sup_{r \to 0} 4\Delta_f(M)/r^2.\]

\(\Lambda_f(M)\) and \(\Lambda^s_f(M)\) are defined likewise, with \(\lim\) and \(\lim\) inf in place of \(\lim\) sup.

(3) Theorem. Suppose

(i) the function \(u\) is upper semi-continuous in the finite plane domain \(D\);
(ii) \(\Lambda^s u(P) \geq 0\) for \(P\) in \(D - E\), where \(E\) is at most countable;
(iii) \(\lim \sup_{r \to 0} \Delta u(P)/r \geq 0\) for \(P\) in \(E\).

Then \(u\) is subharmonic in \(D\).

The special case of the theorem in which \(E\) is vacuous is known \([1, p. 14]\). At the end of the present paper, (3) is used to extend the results of \([3]\) and \([4]\).

Proof of (3). The existence of \(\Lambda^s u(P)\) implies that \(u \in L\) on every sufficiently small circle about \(P\). Hence condition (c) is satisfied. By (ii) and (iii), condition (b) evidently also holds.

It suffices to prove (3) if (ii) is replaced by

(ii') \(\Lambda^s u(P) > 0\) \(\quad (P\) in \(D - E)\).

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1 Numbers in brackets refer to the bibliography at the end of the paper.
For, having established the theorem for this case, we put

$$u_m(P) = u_m(x, y) = u(x, y) + x^2/m \quad (m = 1, 2, 3, \ldots ; P \text{ in } D),$$

so that $\Lambda^* u_m(P) = \Lambda^* u(P) + 2/m > 0 \quad (m = 1, 2, 3, \ldots ; P \text{ in } D - E)$, and $u_m$ satisfies (iii). Letting $m \to \infty$, we see that $u$ is the limit of a monotonically decreasing sequence of subharmonic functions, and is therefore itself subharmonic [2, p. 335].

We assume (ii') to hold. For $P$ in $D$, we put

$$\mu(u; P) = \text{l.u.b. sup}_{0 \leq \delta < 2\pi} \limsup_{r \to 0} \frac{u(r, \theta) - u(P)}{r}$$

where l.u.b. stands for least upper bound, and $(r, \theta)$ is a polar coordinate system with pole at $P$. If the theorem is false, there exists a domain $G$ with boundary $B$, $G + B \subset D$, and a function $h$, continuous on $G + B$, harmonic in $G$, and such that

$$u(P) \leq h(P) \quad \text{(P on } B),$$

$$u(Q) > h(Q) \text{ for some } Q \text{ in } G.$$  

We may assume that $G$ is bounded. (In fact, we can assume that $G$ is circular [2, p. 334], but this would not simplify the proof.) Furthermore, we can choose $h$ such that

$$|\nabla h(P)| \neq \mu(u; P) \quad \text{(P in } G \cdot E),$$

where $\nabla h$ is the gradient of $h$. For suppose (7) is false. Since $G$ is bounded, we can choose a coordinate system $(x, y)$ such that $x > 0$ for all points of $G$. Clearly there exists $\delta > 0$ such that (5) and (6) hold if $h(P)$ is replaced by $h_1(P) = h(P) + \varepsilon \chi$ (P in $G + B$; $0 \leq \varepsilon \leq \delta$). Since $E$ is at most countable, we can choose $\varepsilon$ in $[0, \delta]$ such that (7) is also satisfied by $h_1$.

Put $w(P) = u(P) - h(P)$. By (5) and (6), the upper semi-continuous function $w$ attains its maximum at a point $M$ in $G$. It follows that

$$\Lambda^* w(M) = \Lambda^*(w(M) + h(M)) = \Lambda^* w(M) \leq 0,$$

which contradicts (ii') if $M \in D - E$.

If $M \in E$, $\limsup_{r \to 0} \Delta_r w(M) r \geq 0$. Since $w(r, \theta) \leq w(M)$ for sufficiently small $r$, an application of Fatou's theorem gives

$$0 \leq \int_0^{2\pi} \liminf_{r \to 0} \frac{w(M) - w(r, \theta)}{r} \, d\theta \leq \liminf_{r \to 0} \int_0^{2\pi} \frac{w(M) - w(r, \theta)}{r} \, d\theta \leq 0.$$

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It follows that

\[
\limsup_{r \to 0} \frac{w(r, \theta) - w(M)}{r} = 0 \quad \text{for almost all } \theta,
\]

\[
\limsup_{r \to 0} \frac{w(r, \theta) - w(M)}{r} \leq 0 \quad (0 \leq \theta < 2\pi).
\]

Therefore, putting \( \Theta = i \cos \theta + j \sin \theta \), where \( i, j \) are unit vectors in the \( x, y \) directions, we have

\[
\limsup_{r \to 0} \frac{u(r, \theta) - u(M)}{r} = \Theta \cdot \nabla h(M) \quad \text{for almost all } \theta,
\]

\[
\limsup_{r \to 0} \frac{u(r, \theta) - u(M)}{r} \leq \Theta \cdot \nabla h(M) \quad (0 \leq \theta < 2\pi).
\]

Hence \( \mu(u; M) = |\nabla h(M)| \), which contradicts (7). This completes the proof.

The theorem just proved enables us to derive the conclusion of Theorem I of [3] from weaker hypotheses. We obtain the following theorem.

(8) Theorem. Let \( F \) be continuous in the finite plane domain \( D \). Let \( Z \) be a closed and bounded set of capacity zero. Suppose

(i') \( \Lambda^* F(P) > -\infty \) on \( D - D \cdot Z - E_1 \), where \( E_1 \) is at most countable, and \( \limsup_{r \to 0} \frac{\Lambda r F(P)}{r} \geq 0 \) on \( E_1 \);

(ii') \( \Lambda^* F(P) < +\infty \) on \( D - D \cdot Z - E_2 \), where \( E_2 \) is at most countable, and \( \liminf_{r \to 0} \frac{\Lambda r F(P)}{r} \leq 0 \) on \( E_2 \);

(ii) there exists a function \( y \), defined in \( D \), such that \( y \in L \) on every closed subset of \( D \), and such that \( y(P) \leq \Lambda^* F(P) \) for \( P \) in \( D \).

Then, for all bounded domains \( R \) whose closure is contained in \( D \), and at almost all points \( P \) of \( R \),

\[
F(P) = \frac{1}{2\pi} \int \int_R \Lambda F(Q) g(P, Q) dQ + H(P),
\]

where \( g(P, Q) \) is Green's function for \( R \), and \( H \) is harmonic in \( R \) and assumes the values of \( F \) on the boundary of \( R \).

In [3], this theorem is proved for the case \( E_1 = E_2 = 0 \). To prove (8), we use the notation

\[
\Omega f(P) = \frac{1}{2\pi} \int \int_R f(Q) g(P, Q) dQ,
\]

as in [3], and put
where \( u \) is an upper semi-continuous function associated with \( y \) in the sense of the Vitali-Carathéodory theorem. As in (4.2.2) of [3], we have now

\[
\Lambda^* W(P) \geq 0 \quad (P \in R - R \cdot Z - R \cdot E_A).
\]

Since \( u \) is bounded above in \( R \), we have, by (3.5.1) of [3],

\[
\limsup_{r \to 0} \frac{\Delta_r \Omega u(P)}{2r} = \limsup_{r \to 0} \frac{1}{2 \pi r} \int \int_{PQ < r} u(Q) \log \frac{r}{PQ} dQ 
\leq \limsup_{r \to 0} \frac{mr}{4} = 0 \quad (P \in R),
\]

where \( m \) is an upper bound of \( u \) in \( R \). Hence

\[
\limsup_{r \to 0} \frac{1}{r} \Delta_r W(P) 
\geq \limsup_{r \to 0} \frac{1}{r} \Delta_r F(P) - \limsup_{r \to 0} \frac{1}{r} \Delta_r \Omega u(P) \geq 0
\]

for \( P \) on \( R - E \). By (9), (11), and (3), \( W \) is subharmonic in the domain \( R - R \cdot Z \). The proof then proceeds as in 4.2 of [3].

The hypotheses of Theorem II of [3] and of Theorems 2.6, 3.9, and 4.2 of [4] may be weakened in the same manner.

Finally, we wish to state the following analogue of (3).

\[
\textbf{(12) Theorem. Suppose}
\]

(i) the function \( f \) is upper semi-continuous in the segment \( I = (a, b) \);

(ii) \( \limsup_{h \to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} \geq 0 \) for \( x \) in \( I - E \), where \( E \) is at most countable;

(iii) \( \limsup_{h \to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} \geq 0 \) for \( x \) in \( E \).

Then \( f \) is convex in \( (a, b) \).

\[\text{Bibliography}\]


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