The determination of the value and optimal strategies of a zero-sum, two-person game with a finite number of pure strategies can be a lengthy process, involving, among other things, the calculation of the value

$$\sup_x \inf_y (xB, y),$$

where $B$ is a real matrix with $m$ rows and $n$ columns, $x$ ranges over the set of row vectors with $m$ components, all non-negative and adding up to one, $y$ ranges over the corresponding set of $n$-component column vectors, and the pay-off, $(xB, y)$, indicates the inner product of the two vectors $xB$ and $y$. One device which may simplify a game computation is that of “dominance” or “majorization” \[vNM, p. 174\] by which the solution of a game is reduced to the solution of a smaller game, that is, one with a smaller number of pure strategies. There is another device which, when conditions are right, may simplify the solution of a game by reducing it to the solution of smaller games. This device, presented here, gives either the value or a bound for it, depending on the information available about the sub-games. It also gives an optimal strategy or a strategy sufficient to insure an outcome not worse than that predicted by the aforementioned bound. It is particularly effective when there are rows (or columns) in $B$, which are constant or have large constant segments.

Let $B$ be a game matrix (rows maximizing) decomposed into

$$B = \{ B_{ij} \mid 1 \leq i \leq M, 1 \leq j \leq N \},$$

where $B_{ij}$ is a sub-matrix with $m_i$ rows and $n_j$ columns (the $m_i$ rows being independent of $j$ and the $n_j$ columns being independent of $i$). Let the value of $B$ be $v$ and the value of $B_{ij}$ be $v_{ij}$. Let the set of optimal strategies for the first player in the game be $X = \{ x \}$ and the set of optimal strategies for the first player in the sub-game be $X_{ij} = \{ x^i \}$. Let $Y$ and $Y_{ij}$ represent the corresponding sets for the

1 This originally appeared in a RAND report: Total reconnaissance with total countermeasures: Simplified model, August 5, 1949, P-106, Rand Corporation, Santa Monica, California. For the definitions in game theory see [vNM]. See the bibliography at the end of the paper.
second players. Let $\mathbf{B}$ be the $M \times N$ matrix with entries $v_{ij}$. Let $\hat{x} = \{x_i | 1 \leq i \leq M, x_i \geq 0, \sum x_i = 1\}$ be a typical optimal strategy for the first player in the game with matrix $\mathbf{B}$, $\hat{X}$ the set of optimal strategies for the first player in $\mathbf{B}$, and $\hat{y}$ and $\hat{Y}$ the analogous items for the second player. Let $\hat{v}$ be the value of the game with matrix $\mathbf{B}$.

**Theorem.** $\bigcap_i X_i^* \neq \emptyset$ for each $i$ implies $v \geq \hat{v}$. If $x^i \in \bigcap_i X_i^*$ for each $i$ and $\hat{x} \in \hat{X}$, then by playing the vector $\{\hat{x}_1 x^1, \hat{x}_2 x^2, \cdots, \hat{x}_M x^M\}$ (where by this notation we mean the vector each of whose first $m_i$ components are $\hat{x}_i$ multiplied by the appropriate one of the $m_i$ components of $x^i$, and so on) the first player may assure himself of a pay-off of at least $\hat{v}$.

**Proof.** Let a typical strategy for II in game with matrix $\mathbf{B}$ be

$$y = \{\beta^1 \hat{y}_1, \cdots, \beta^N \hat{y}_N\}$$

where $\hat{y}_j$ is a vector with $n_j$ non-negative components adding up to one and $\beta^j \geq 0$ for each $j$, $\sum \beta^j = 1$. If I plays

$$\{\hat{x}_1 x^1, \cdots, \hat{x}_M x^M\}$$

then the pay-off

$$\sum_{j=1}^N \left( \sum_{i=1}^M \hat{x}_i x^i B^i_{ji} \beta^j \hat{y}_j \right) \geq \sum_{j=1}^N \left( \sum_{i=1}^M \hat{x}_i \beta^j \right) \geq \hat{v}.$$

**Corollary.** $\bigcap_j Y_j^* \neq \emptyset$ for each $j$ implies $v \leq \hat{v}$. If $\hat{y}_j \in \bigcap_j Y_j^*$ for each $j$ and $\hat{y} \in \hat{Y}$, then by playing the vector $\{\hat{y}_1 y_1, \cdots, \hat{y}_N y_N\}$ the second player may limit his losses to $\hat{v}$.

**Corollary.** $\bigcap_i X_i^* \neq \emptyset$ for each $i$ and $\bigcap_j Y_j^* \neq \emptyset$ for each $j$ implies $v = \hat{v}$. $x^i \in \bigcap_i X_i^*$, $y_j \in \bigcap_j Y_j^*$, $\hat{x} \in \hat{X}$ and $\hat{y} \in \hat{Y}$ implies $\{x_1 x^1, \cdots, x_M x^M\} \in \hat{X}$ and $\{\hat{y}_1 y_1, \cdots, \hat{y}_N y_N\} \in \hat{Y}$.

**Bibliography**


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\* Similar results (unpublished) have been obtained by Gale, Kuhn, and Tucker independently of those of the author. An abstract, apparently motivated by consideration of matrices $B$ which have large constant segments, of these results is D. Gale, H. W. Kuhn, and A. W. Tucker, Bull. Amer. Math. Soc. Abstract 55-11-472.