THE RANGE OF CERTAIN VECTOR INTEGRALS

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1. **Introduction.** Let $u_1, \cdots, u_n$ be completely additive set functions defined over a Borel field $\mathcal{B}$ of subsets of a space X, and let A be any bounded subset of Euclidean n-space. With every $\mathcal{B}$-measurable function $f=a(x)=[a_1(x), \cdots, a_n(x)]$ defined on X with range in A we associate the vector $v(f)=(\int a_1(x)du_1, \cdots, \int a_n(x)du_n)$. Our problem is to investigate the range R of the function $v(f)$.

If A consists of the two points $(0, 0, \cdots, 0)$ and $(1, 1, \cdots, 1)$, the functions f have the form $\phi(x), \cdots, \phi(x)$, where $\phi(x)$ is the characteristic function of a $\mathcal{B}$-measurable set E, and the range R of $v(f)$ is the range of the vector measure $u_1(E), \cdots, u_n(E)$, $E \in \mathcal{B}$. This case has been treated by Liapounoff[1] who has shown that R is closed and, if $u_1(E), \cdots, u_n(E)$ are nonatomic, convex. A simplified proof of Liapounoff's results has been given by Halmos.[2]

Our results are extensions of those of Liapounoff. We shall show (1) that whenever A is closed, R is closed, and (2) that whenever $u_1, \cdots, u_n$ are nonatomic, R is convex. As will be noted below, result (2) follows directly from the corresponding result of Liapounoff; nevertheless we give an independent proof, as our methods differ in detail, though not in essential idea, from those of Halmos, and the principal tool, Theorem 2, is a curious result of some interest.

We shall sketch here the application of the results to statistical decisions and the theory of games, in the special case where $u_1, \cdots, u_n$ are probability measures. Nature (Player I) chooses an integer $i=1, \cdots, n$, a point x is then chosen from X according to the distribution u_i, and the statistician (Player II) observes x. He then choose a point $a=(a_1, \cdots, a_n) \in A$ and loses the amount a_i. A strategy for the statistician is a function $f=a(x)$, specifying for each $x \in X$ the point a to be chosen when x is observed, and the vector $v[f]$ is his expected loss vector; its ith coordinate is the statistician's expected loss when Nature chooses i and he uses strategy f. Thus R is the set of loss vectors the statistician can achieve. Now by a mixed strategy, that is, using N strategies $f_1, \cdots, f_N$ with specified probabilities $\lambda_1, \cdots, \lambda_N, \lambda_i \geqq 0, \sum \lambda_i = 1$, the statistician can achieve

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[1] *Sur les fonctions-vecteurs complètement additives*, Bull. Acad. Sci. URSS. Sér. Math. vol. 4 (1940) pp. 465–478.

[2] *The range of a vector measure*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 416–421.

as an expected loss vector precisely the points in the convex set determined by R. Thus whenever R is convex, any vector which can be obtained with a mixed strategy can already be obtained with a pure strategy; mixed strategies are unnecessary. Closure of R alone seems to have no particular game theory significance, but if R is convex and closed, it follows from a theorem of Wald[3] that the statistician has a good pure strategy. Our results, then, have the following consequences: if $u_1, \cdots, u_n$ are nonatomic, mixed strategies are unnecessary; if in addition A is closed, the statistician has a good pure strategy.[4]

2. **Separation of atomic and nonatomic cases.** Let u be any nonnegative measure such that each u_i is absolutely continuous with respect to u; there are functions $p_1(x), \cdots, p_n(x)$ such that $\int_E p_i(x)du(x) = u_i(E)$, $i = 1, \cdots, n$. Let $X = X_1 + X_2$, where u is nonatomic on X_1, that is, every set of positive u-measure contains a set of every smaller positive measure, and completely atomic on X_2, that is, $X_2 = S_1 + S_2 + \cdots$, and every $\mathcal{B}$-measurable subset S of S_i has $u(S) = 0$ or $u(S) = u(S_i)$.[5] Now the range R of $v[f]$ is simply the vector sum of R_1, R_2, the ranges obtained when $X, u_1, \cdots, u_n$ are contracted to X_1, X_2. Thus we need consider only the two cases u nonatomic, u completely atomic.

3. **Atomic case.**

THEOREM 1. *If u is atomic and A is closed, then R is closed.*

PROOF. If $X = S_1 + S_2 + \cdots$ is the decomposition of X into atoms, every $\mathcal{B}$-measurable function is constant almost everywhere over each S_i; we specify the function by the sequence $\{v_i\}$, where v_i is its value almost everywhere on S_i. The admissible functions f are sequences $\{v_i\}$ of points of A. Then every sequence $\{f_j\}$ of admissible f's has a subsequence converging almost everywhere to a function f^* (A is bounded). Since A is closed, the values of f^* are in A so that f^* is admissible. If $v(f_j) \to \xi$, then $v(f^*) = \xi$, so that R is closed.

4. **Nonatomic case: convexity of R.**

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[3] *Generalization of a theorem by von Neumann concerning zero sum two person games*, Ann. of Math. vol. 46 (1945) pp. 281–286.

[4] A more detailed discussion and related results are given in Dvoretzky, Wald, and Wolfowitz, *Elimination of randomization in certain problems of statistics and of the theory of games*, Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) pp. 256–260, and in a note *On a theorem of Liapounoff*, submitted by the author to the Ann. Math. Statist.

[5] That such a decomposition exists is well known; see for instance the author's *Idempotent Markoff chains*, Ann. of Math. vol. 43 (1942) pp. 560–567.

LEMMA. *If u is any nonatomic, non-negative measure on a Borel field $\mathcal{B}$ of subsets of a space X, and $f(x)$ is any u-integrable function, there is a set S with $u(S)=(1/2)u(X)$, $\int_S f(x)du=(1/2)\int f du$.*

PROOF. We merely outline the construction of such a set S; the details are straight forward. Let a^* be the maximum of all real numbers a for which $u\{f\geqq a\}\geqq(1/2)u(X)$. For every $a\leqq a^*$ there are sets S with $u(S)=(1/2)u(X)$ of the form $U+V+W$, where $U=\{a<f(x)<b\}$, V is a subset of $\{f(x)=a\}$, W is a subset of $\{f=b\}$. Let $\phi(S)=\int_S f(x)du$. If $a_1<a_2\leqq a^*$, for any S_1, S_2 corresponding to a_1, a_2, $\phi(S_1)\leqq\phi(S_2)$. The minimum value of $\phi(S)$ occurs with $a=-\infty$, V equal to a null set; the maximum value of $\phi(S)$ occurs at $a=a^*$, $b=+\infty$, W equal to a null set; and $(1/2)\int f(x)du=\alpha$ lies between these two values. For an a with $u\{f=a\}=0$, there is only one value of $\phi(S)$, while if $u\{f=a\}>0$, $\phi(S)$ increases continuously as $u(V)$ decreases from $u\{f=a\}$ to 0, except at $a=a^*$, where $u(V)$ may have a positive minimum. If $\psi_1(a)$, $\psi_2(a)$ are the minimum and maximum values of $\phi(S)$ for a given a, ψ_1, ψ_2 are continuous from the left and right respectively. If a_0 is the maximum a with $\psi_1(a)\leqq\alpha$, we have, for $a>a_0$, $\psi_2(a)\geqq\psi_1(a)>\alpha$, so that, $\psi_2(a)$ being continuous from the right, $\psi_2(a_0)\geqq\alpha$. Since for S corresponding to a_0 we can make $\phi(S)$ assume any value between $\psi_1(a_0)$, $\psi_2(a_0)$, there is an S corresponding to a_0 with $\phi(S)=\alpha$.

THEOREM 2. *If u is any nonatomic, non-negative measure on a Borel field $\mathcal{B}$ of subsets of X and $f_1, \cdots, f_n$ are any $\mathcal{B}$-measurable functions with $\int f_i du$ finite, $i=1, \cdots, n$, there is a Borel field $\mathcal{A}\subset\mathcal{B}$ such that u is nonatomic on $\mathcal{A}$ and for every $D\in\mathcal{A}$,*

$$\int_D f_i(x)du = u(D)\int f_i(x)du.$$

PROOF. It is no loss of generality to suppose $u(X)=1$ and use the language of probability. The theorem then asserts that $E_{\mathcal{A}}(f_i)=E(f_i)$, $i=1, \cdots, n$, where $E(f)$, $E_{\mathcal{A}}(f)$denote the expectation and conditional expectation with respect to the Borel field $\mathcal{A}$ of the chance variable f.

We first prove the theorem for the single chance variable f_1. Suppose $E(f_1)=\alpha$. According to the lemma there is a set S, with $u(S)=1/2$, $\int_S f_1 du=(1/2)\alpha$. Applying the lemma to the sets $S=S_1$, $CS=S_2$, and f_1, we obtain four disjoint sets S_{11}, S_{12}, S_{21}, S_{22} such that $S_{i1}+S_{i2}=S_i$, $u(S_{ij})=1/4$, $\int_{S_{ij}} f_1 du=(1/4)\alpha$. Continuing in this way, and denoting the Borel field determined by the 2^n sets obtained at the nth stage by $\mathcal{B}_n$, we see that $\mathcal{B}_1\subset\mathcal{B}_2\subset\cdots$, $E_{\mathcal{B}_n}(f_1)=E(f_1)$, and

$\mathcal{B}_n$ contains 2^n disjoint sets of measure $(1/2)^n$. If $\mathcal{A}_1$ is the smallest Borel field containing $\mathcal{B}_1, \mathcal{B}_2, \cdots, u$ is nonatomic on $\mathcal{A}_1$ and, according to a theorem of Doob,[6] $E_{\mathcal{B}_n}(f_1) \to E_{\mathcal{A}_1}(f_1)$. Thus $E_{\mathcal{A}_1}(f_1) = E(f_1)$. Now applying the result for a single function to $E_{\mathcal{A}_1}(f_2)$ with $\mathcal{B}$ replaced by $\mathcal{A}_1$, we obtain a Borel field $\mathcal{A}_2 \subset \mathcal{A}_1$ on which u is nonatomic, and on which $E_{\mathcal{A}_2}[E_{\mathcal{A}_1} f_2] = E[E_{\mathcal{A}_1} f_2]$. Since $E_{\mathcal{A}_2}[E_{\mathcal{A}_1} f] = E_{\mathcal{A}_2}(f)$ and $E[E_{\mathcal{A}_1} f] = E(f)$ for any f, we have $E_{\mathcal{A}_2}(f_i) = E(f_i)$, $i = 1, 2$. Continuing in this way, we obtain a decreasing sequence of Borel fields $\mathcal{A}_1 \supset \mathcal{A}_2 \supset \cdots \supset \mathcal{A}_n$ such that $\mathcal{A}_n = \mathcal{A}$ has the property asserted in the theorem.

THEOREM 3. *If u is nonatomic, R is convex.*

PROOF. Suppose $v[f_1] = r_1$, $v(f_2) = r_2$, where $f_1 = (a_{11}(x), \cdots, a_{1n}(x))$, $f_2 = (a_{21}(x), \cdots, a_{2n}(x))$. Let $\mathcal{A}$ be a Borel field such that u is nonatomic on $\mathcal{A}$, and $E_A(g) = E(g)$ for $g = a_{11}p_1, \cdots, a_{1n}p_n, a_{21}p_1, \cdots, a_{2n}p_n$. Let D be a set in $\mathcal{A}$ with $u(D) = t$, $0 \leqq t \leqq 1$, and define $f = f_1$ on D, f_2 on CD. Then

$$v(f) = \left(\int_D a_{11}p_1 du + \int_{CD} a_{21}p_1 du, \cdots, \int_D a_{1n}p_n du + \int_{CD} a_{2n}p_n du \right)$$
$$= tr_1 + (1 - t)r_2.$$

As remarked above, Theorem 3 is a direct consequence of the theorem of Liapounoff: we consider the $2n$-dimensional measure $w(D) = \int_D a_{11}p_1 du, \cdots, \int_D a_{1n}p_n du, \int_D a_{21}p_1 du, \cdots, \int_D a_{2n}p_n du$. We have $w(X) = (r_1, r_2)$, so that for any t, $0 \leqq t \leqq 1$, by Liapounoff's theorem, there is a D with $w(D) = (tr_1, tr_2)$. Then $w(CD) = [(1-t)r_1, (1-t)r_2]$ and, defining $f = f_1$ on D, $f = f_2$ on CD, we obtain $v(f) = tr_1 + (1-t)r_2$.

5. Nonatomic case, A closed: closure of R.

THEOREM 4. *Suppose A is closed, and let $L_1, \cdots, L_k$ be any linear functions on n-dimensional space. Let $\bar{R}$ be the closure of R; define $\lambda_1 = \min_{r \in \bar{R}} L_1(r)$, $S_1 = \bar{R}\{L_1(r) = \lambda_1\}$, and, inductively for $1 < i \leqq k$, $\lambda_i = \min_{r \in S_{i-1}} L_i(r)$, $S_i = S_{i-1}\{L_i(r) = \lambda_i\}$. Then there is a point $r \in R$ with $L_i(r) = \lambda_i$, $i = 1, \cdots, k$.*

PROOF. We may suppose, choosing additional L_i if necessary, that there are n linearly independent linear functions among the L_i. Let r^* be the point such that $L_i(r^*) = \lambda_i$, $i = 1, \cdots, k$. Then $r^* \in \bar{R}$. Let $\{r_j\}$ be a sequence of points in R with $r_j \to r^*$; say $r_j = v(f_j)$, $f_j = [a_{1j}(x),$

[6] *Regularity properties of certain families of chance variables*, Trans. Amer. Math. Soc. vol. 47 (1940) p. 460, Theorem 1.3.

$\cdots, a_{nj}(x)]$. Define $\phi_{ij}(x) = L_i[a_{1j}(x)p_1(x), \cdots, a_{nj}(x)p_n(x)]$. Then $\int\phi_{ij}(x)du \to \lambda_i$ as $j\to\infty$. We shall show that there is a subsequence j_t such that $\phi_{ij_t}(x)\to\phi_i(x)$ as $t\to\infty$ except on an x-set of u-measure zero.

Define

$$\phi_N(x) = \min\,[\phi_{11}(x), \cdots, \phi_{1N}(x)],$$
$$S_{kN} = \{\phi_{1j} > \phi_N \text{ for } j < k,\ \phi_{1k} = \phi_N\},$$
$$f_N^* = f_k \text{ on } S_{kN}.$$

Then[7] $L_1(f_N^*) = \int\phi_N(x)du$. Now $\phi_N(x) \leqq \phi_{1N}(x)$, so that $L_1(f_N^*) \leqq L_1(f_N)$, $L_1(f_N^*)\to\lambda_1$. Let $\phi(x) = \lim \phi_N(x)$, $N\to\infty$. Then $\phi_{1N} \geqq \phi$ for all x and $\int\phi_{1N}du\to\int\phi du$. Consequently $\phi_{1N}\to\phi$ in u-measure, and there is a subsequence of $\phi_{1N}\to\phi$ almost everywhere. Suppose we have found a subsequence of j for which ϕ_{ij} converges almost everywhere to a function $\phi_i(x)$ for $1\leqq i<m$. To simplify notation, suppose the original sequence ϕ_{ij} has this property. If ϕ_{mj} does not converge in u-measure, there is an $\epsilon>0$ and a sequence of integers $s_1, t_1, s_2, t_2, \cdots, s_j, t_j$ becoming infinite with j, for which the set $T_j = \{\phi_{ms_j} - \phi_{mt_j} > \epsilon\}$ has $u(T_j) > \epsilon$. Define $g_j = f_{mt_j}$ on T_j, $g_j = f_{ms_j}$ on $C(T_j)$. Then $L_i(g_j)\to\lambda_i$ for $i<m$, and $L_m(g_j) \leqq L_m(f_{s_j}) - \epsilon^2$, so that lim sup $L_m(g_j) \leqq \lambda_m - \epsilon^2$. Since $v_j = v(g_j)\in R$, there is a subsequence of v_j approaching $v_0 \in \overline{R}$. We have $L_i(v_0) = \lambda_i$ for $i<m$, $L_m(v_0) \leqq \lambda_m - \epsilon^2$, contradicting the definition of λ_m. Hence ϕ_{mj} converges in u-measure to a limiting function $\phi_m(x)$ and there is a subsequence converging almost everywhere to $\phi_m(x)$. This completes the induction.

We may now suppose, replacing f_j by an appropriate subsequence, that $\phi_{ij}(x)\to\phi_i(x)$ almost everywhere as $j\to\infty$ for $i=1, \cdots, k$; that is, $L_i[a_{1j}p_1, \cdots, a_{nj}p_n]\to\phi_i(x)$, $i=1, \cdots, k$. Since $L_1, \cdots, L_k$ contain n linearly independent functions, the sequence of points $\{(a_{1j}p_1, \cdots, a_{nj}p_n)\}$ converges almost everywhere to a limiting point $w(x) = (w_1(x), \cdots, w_n(x))$. If $p_1, \cdots, p_n$ never vanish, this implies $a_j(x)$ converges to a limiting function $a^*(x)$ almost everywhere; since A is closed, the values of $a^*(x)$ are in A, $f = a^*(x)$ is an admissible function, and $L_i(f) = \lim L_i(f_j) = \lambda_i$, $i=1, \cdots, n$, and the proof is complete. If $p_1, \cdots, p_n$ sometimes vanish, we make the following modification. We need only find an $a^*(x) = (a_1^*(x), \cdots, a_n^*(x))$ such that the tth component $a_{jt}(x)$ of $a_j(x)$ converges to $a_t^*(x)$ for all x with $p_t(x)\neq 0$; the values of $a_t^*(x)$ on $\{p_t = 0\}$ do not influence $v[a^*]$. For any subset $\alpha = (i_1, \cdots, i_c)$ of $(1, \cdots, n)$, let U_α be the x-set where $p_t \neq 0$ for $t\in\alpha$, $p_t = 0$ for $t\notin\alpha$. Let h_α be a Baire func-

[7] For brevity we write $L_i(f)$ for $L_i[v(f)]$.

tion mapping c-dimensional space into n-dimensional space in such a way that every point $(x_1, \cdots, x_c)$ in the projection of A on the c-dimensional subspace $(a_{i_1}, \cdots, a_{i_c})$, that is, every point $(x_1, \cdots, x_c)$ for which there is a point $a=(a_1, \cdots, a_n)\in A$ with $a_{i_1}=x_1, \cdots, a_{i_c}=x_c$, has $h_\alpha(x_1, \cdots, x_c)=a^*=(a_1^*, \cdots, a_n^*)\in A$, with $a_{i_1}^*=x_1, \cdots, a_{i_c}^*=x_c$.[8] On U_α, $a_{jt}(x)$ converges for $t\in\alpha$, say to $a_t^*(x)$. Then $a^*(x)=h_\alpha[a_{i_1}^*(x), \cdots, a_{i_c}^*(x)]$ on U_α is the required function.

THEOREM 5. *If u is nonatomic and A is closed, then R is closed.*

PROOF. According to Theorem 3, R is convex. With the property of R proved in Theorem 4, its closure follows from the following fact about convex sets.

LEMMA. *Let B be any closed bounded convex set, and let D be any convex set with the property* (*): *For any linear functions $L_1, \cdots, L_k$, if we define $\lambda_1=\min_{x\in B}L_1(x)$, $B_1=B\{L_1(x)=\lambda_1\}$ and, inductively for $1<i\leqq k$, $\lambda_i=\min_{x\in B_{i-1}} L_i(x)$, $B_i=B_{i-1}\{L_i(x)=\lambda_i\}$, there is a point $d\in BD$ with $L_i(d)=\lambda_i$, $i=1, \cdots, k$. Then D contains B.*

This is easily established by induction on the dimension of B. If B is one-dimensional, that is, a closed interval, D must contain the end points of B so that $D\supset B$. Suppose the lemma established for sets B of dimension less than s, and let B be s-dimensional. The intersection B_1 of B with any supporting hyperplane $L_1(x)=\lambda_1$, where $\lambda_1=\min_{x\in B} L_1(x)$, is a closed convex set of dimension less than s and has the property (*) relative to D. By the induction hypothesis, $D\supset B_1$. Thus D contains the intersection of B with every supporting hyperplane, so that $D\supset B$.

THEOREM 6. *If A is closed, R is closed.*

PROOF. This follows from Theorems 1 and 5, by splitting u into atomic and nonatomic parts with ranges R_1, R_2 and noting that R is the vector sum of the closed sets R_1, R_2.

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[8] Such a Baire function may be constructed as follows: To simplify notation, say $\alpha=(1, \cdots, c)$. Let A_d be the projection of A on its first d coordinates, $c\leqq d\leqq n$. Then each A_d is closed and is the projection of A_{d+1} on its first d coordinates. If we can construct a Baire function h_d mapping d-space into $d+1$-space with the required property, where A is replaced by A_{d+1}, the Baire function $h_{n-1}[h_{n-2}[\cdots[h_c(x_1, \cdots, x_c)]\cdots]]$ is the required function. Thus we may suppose $c=n-1$. The function $\psi(x_1, \cdots, x_{n-1})=\min_{(x_1, \cdots, x_{n-1}, a_n)\in A} a_n$ on A_{n-1}, and 0 elsewhere, is lower semi-continuous on the closed set A_{n-1} and so is a Baire function, and the function $h(x_1, \cdots, x_{n-1})=[x_1, \cdots, x_{n-1}, \psi(x_1, \cdots, x_{n-1})]$ is the required function.