

$$(13) \quad c_q(A_{qij} - A_{qji}) = 0$$

is a necessary condition for the permanence of vector-lines. It is also sufficient, as we see from consideration of the order of the equations involved and the fact that if (13) holds, then $\Omega_{ij} = 0$ implies $\partial\Omega_{ij}/\partial t = 0$.

Hence we have this result: *A necessary and sufficient condition for the permanence of the vector-lines of c_i is (13), or equivalently, that the tensor $c_q A_{qij}$, that is,*

$$c_j(c_{i,k}v_k + \partial c_i/\partial t) + c_i v_{j,k}c_k,$$

shall be symmetric.

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NOTE ON A THEOREM IN SUMMABILITY

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Let T denote a regular matrix method of summability in the complex domain, that is to say, a transformation of the form

$$t_n = \sum_{k=1}^{\infty} a_{nk} s_k \quad (n = 1, 2, 3, \dots),$$

having the property that the convergence of $\{s_k\}$ to s always implies the existence of t_n for each n and the convergence of $\{t_n\}$ to s . It is well known that the following conditions of Silverman-Toeplitz are necessary and sufficient in order that T be regular: $a_{nk} = o(1)$ ($n \rightarrow \infty$; $k = 1, 2, 3, \dots$); $\sum_{k=1}^{\infty} a_{nk} = 1 + o(1)$ ($n \rightarrow \infty$); and

$$(1) \quad \sum_{k=1}^{\infty} |a_{nk}| = O(1) \quad (n \rightarrow \infty).$$

The following theorem was established recently by Henstock [2].¹

THEOREM (HENSTOCK). *Let $y \equiv \{z_k\}$ be a given bounded sequence of complex numbers. Then there exist denumerably many sequences of*

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¹ Numbers in brackets refer to the references at the end of the paper.

zeros and ones, depending only on y , such that the summability of each of these by any regular matrix method T implies the T -summability of y .

The proof given by Henstock follows classical lines and is moderately complicated. The purpose of this note is to give a shorter and simpler proof based on the theory of linear operations [1]. To this end we consider the sequences $X(t) \equiv \{\alpha_k\}$ of zeros and ones, with infinitely many ones, where $t \equiv .\alpha_1\alpha_2\alpha_3 \cdots$ (radix 2). It is known [3] that the set of all sequences $X(t)$ constitutes a fundamental set in the Banach space (b) of all bounded complex sequences $x \equiv \{s_k\}$ with $\|x\| \equiv \sup_k |s_k|$. In other words, for $p = 1, 2, 3, \cdots$ and an arbitrarily given $y \in (b)$ there exist a finite number of complex constants $A_1^p, A_2^p, \cdots, A_{m_p}^p$ and corresponding sequences $X(t_1^p), X(t_2^p), \cdots, X(t_{m_p}^p)$ such that $\|y - [A_1^p X(t_1^p) + A_2^p X(t_2^p) + \cdots + A_{m_p}^p X(t_{m_p}^p)]\| < 1/p$. If y_p denotes the sequence in brackets we therefore have $y_p \rightarrow y$ in the norm of (b) . Let T be a regular matrix method and write $f_n(x) = \sum_{k=1}^{\infty} a_{nk} s_k$ ($n = 1, 2, 3, \cdots$) where $x = \{s_k\} \in (b)$. For each n , $f_n(x)$ is a linear operation on (b) to the space of complex numbers, and $\|f_n\| = \sum_{k=1}^{\infty} |a_{nk}|$ [3]. If each of the denumerably many sequences $X(t_i^p)$ ($i = 1, 2, \cdots, m_p; p = 1, 2, 3, \cdots$) is summable- T , then the sequence y_p is summable- T , that is, the sequence $\{f_n(y_p)\}$ is convergent for each p . Since the sequence of norms $\{\|f_n\|\}$ is bounded by (1) and $y_p \rightarrow y$, it follows (as in [1, p. 79, proof of Theorem 3]) that the sequence $\{f_n(y)\}$ is convergent. This completes the proof.

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3. H. J. Hamilton and J. D. Hill, *Operation theory and multiple sequence transformations*, Duke Math. J. vol. 8 (1941) pp. 154-162.

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