NOTES ON FOURIER ANALYSIS. XXX ON THE ABSOLUTE CONVERGENCE OF CERTAIN SERIES OF FUNCTIONS[1]

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1. **Introduction.** Recently O. Szász [4][2] has proved the following theorems which are generalizations of Fatou's theorem (cf. Zygmund [5, p. 132]).

THEOREM A. *If the series*

$$\sum_{n=1}^{\infty} a_n \cos nx, \ |a_{n+1}| \leqq c|a_n| \ (c>0); n=1, 2, \cdots),$$

is absolutely convergent at a point x_0, *then* $\sum_{n=1}^{\infty} |a_n| < \infty$. *The same is true for the series* $\sum_{n=1}^{\infty} a_n \sin nx$, *provided that* $x_0 \not\equiv 0 \pmod{\pi}$.

THEOREM B. *Let* $\phi(x)$ *be Riemann integrable in the interval* (0, 1) *and periodic with period* 1 *and* $\int_0^1 |\phi(x)| dx \neq 0$. *If the series*

$$\sum_{n=1}^{\infty} a_n\phi(nx), \ 0<|a_{n+1}|<(1+c/n)|a_n| \ (c>0; \ n=1, 2, \cdots),$$

is absolutely convergent at an irrational point x_0, *then* $\sum_{n=1}^{\infty} |a_n| < \infty$.

There are some gaps between the conditions for the sequences of coefficients $\{a_n\}$ in these two theorems. In §2, we shall prove that the conditions for $\{a_n\}$ in Theorem B can be replaced by those in Theorem A.

On the other hand Reves and Szász [2] have proved the analogues of Cantor-Lebesgue's theorem and Denjoy-Lusin's theorem for the double trigonometric series. In §3, we shall generalize these theorems: the former is on the line of Mazur-Orlicz' generalization for Cantor-Lebesgue's theorem [1, Theorem 1 and its corollary] and the latter is in the direction of Salem's generalization for Denjoy-Lusin's theorem [3, Theorem X].

2. **Theorems of Fatou-Szász type.** We shall give here the generalizations of Theorems A and B.

THEOREM 1. *Let* $\phi(x)$ *be defined in the interval* (0, 1) *and periodic with period* 1 *and suppose that there exists an interval* $I=(a, b)$,

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[1] The authors express their hearty thanks to the referee who gave many valuable suggestions, especially that the original proof of Theorem 1 was incomplete.

[2] Numbers in brackets refer to the literature at the end of the paper.

$0 \leqq a < b \leqq 1$, on which $\phi(x)$ is greater than a positive constant d. If the series $\sum_{n=1}^{\infty} a_n\phi(nx)$ $(a_n \geqq 0)$ is absolutely convergent at an irrational point ξ and if $0 < a_{n+1} \leqq ca_n$ $(c > 0;\ n = 1, 2, \cdots)$, then $\sum_{n=1}^{\infty} a_n < \infty$.

For the proof we need the following lemma.

LEMMA 1. *Let J be an interval and θ be an irrational number contained in the interval* (0, 1). *Then for an arbitrary real number α there exists a positive integer N independent of α such that at least one of the numbers $\alpha + n\theta$, $n = 1, 2, \cdots, N$, is contained in the interval J with modulus* 1.

PROOF. We shall prove more precisely that if we denote the number of the points in the sequence $\{\alpha + \nu\theta\}$, $\nu = 1, 2, \cdots, n$, which are contained with modulus 1 in the interval J by $n_J(\alpha)$, then the ratio $n_J(\alpha)/n$ tends independently on α to the length of the interval J as $n \to \infty$; that is, for any given $\epsilon > 0$, there is a positive integer N independent of α such that $|n_J(\alpha)/n - l| < \epsilon$ for $n > N$, where l is the length of the interval J.

To prove this, let $f(x)$ be the function which is obtained by extending the characteristic function of the interval J with period 1. Then there exist two trigonometrical polynomials $p(x)$ and $P(x)$ which satisfy the inequalities

$$p(x) \leqq f(x) \leqq P(x) \qquad \text{in } (0, 1), \tag{1}$$

$$\int_0^1 [f(x) - p(x)]dx < \epsilon/2, \qquad \int_0^1 [P(x) - f(x)]dx < \epsilon/2. \tag{2}$$

We can write

$$p(x) = \sum_{k=-m}^{m} c_k e^{2\pi ikx}, \qquad P(x) = \sum_{k=-m}^{m} C_k e^{2\pi ikx}. \tag{3}$$

Remembering that

$$\frac{n_J(\alpha)}{n} = \frac{1}{n}\sum_{\nu=1}^{n} f(\alpha + \nu\theta), \tag{4}$$

we have

$$\begin{aligned} \frac{1}{n}\sum_{\nu=1}^{n} p(\alpha + \nu\theta) &\leqq \frac{n_J(\alpha)}{n} \\ &\leqq \frac{1}{n}\sum_{\nu=1}^{n} P(\alpha + \nu\theta) \end{aligned} \tag{5}$$

by (1).

On the other hand

$$
(6)\qquad
\begin{aligned}
\frac{1}{n}\sum_{\nu=1}^{n} p(\alpha+\nu\theta) &= \frac{1}{n}\sum_{\nu=1}^{n}\sum_{k=-m}^{m} c_k e^{2\pi i k(\alpha+\nu\theta)} \\
&= \frac{1}{n}\sum_{\nu=1}^{n}\sum_{k=-m}^{m} c_k e^{2\pi i k\alpha} e^{2\pi i k\nu\theta} \\
&= c_0 + \frac{1}{n}\sum_{k=-m}^{m}{}' c_k e^{2\pi i k\alpha}\sum_{\nu=1}^{n} e^{2\pi i k\nu\theta} \\
&= c_0 + \frac{1}{n}\sum_{k=-m}^{m}{}' c_k e^{2\pi i k\alpha}\,\frac{e^{2\pi i k\theta}-e^{2\pi i k(n+1)\theta}}{1-e^{2\pi i k\theta}},
\end{aligned}
$$

where $\sum'$ denotes the summation omitting the term for $k=0$. Thus we have

$$
(7)\qquad
\begin{aligned}
&\left|\int_0^1 p(x)dx - \frac{1}{n}\sum_{\nu=1}^{n} p(\alpha+\nu\theta)\right| \\
&\qquad\leqq \frac{1}{n}\left|\sum_{k=-m}^{m}{}' c_k e^{2\pi i k\alpha}\,\frac{e^{2\pi i k\theta}-e^{2\pi i k(n+1)\theta}}{1-e^{2\pi i k\theta}}\right| \\
&\qquad\leqq \frac{1}{n}\sum_{k=-m}^{m}{}' |c_k|\,\frac{2}{|1-e^{2\pi i k\theta}|} \\
&\qquad\leqq \frac{2\Delta}{n}\sum_{k=-m}^{m}{}' |c_k|,
\end{aligned}
$$

where

$$
(8)\qquad \Delta = \max_{-m\leqq k\leqq m,\, k\neq 0} \frac{1}{|1-e^{2\pi i k\theta}|}
$$

and Δ is finite since θ is irrational and m is fixed; moreover it is evident from the definition that Δ is independent of α.

Therefore if we choose N_1 such that

$$
(9)\qquad N_1 > 4\Delta\sum_{k=-m}^{m}{}' |c_k|/\epsilon,
$$

we have

$$
(10)\qquad \left|\int_0^1 p(x)dx - \frac{1}{n}\sum_{\nu=1}^{n} p(\alpha+\nu\theta)\right| < \epsilon/2 \qquad \text{for } n > N_1.
$$

Quite similarly we can choose N_2 such that

$$\left| \int_0^1 P(x)dx - \frac{1}{n} \sum_{\nu=1}^{n} P(\alpha + \nu\theta) \right| < \epsilon/2 \qquad \text{for } n > N_2. \tag{11}$$

Let us take $N = \max(N_1, N_2)$, then we have

$$\begin{aligned} \int_0^1 p(x)dx - \epsilon/2 &< \frac{1}{n} \sum_{\nu=1}^{n} p(\alpha + \nu\theta) \leqq \frac{n_J(\alpha)}{n} \\ &\leqq \frac{1}{n} \sum_{\nu=1}^{n} P(\alpha + \nu\theta) < \int_0^1 P(x)dx + \epsilon/2 \end{aligned} \tag{12}$$

from (5), (10), and (11), and finally

$$\int_0^1 f(x)dx - \epsilon \leqq \frac{n_J(\alpha)}{n} \leqq \int_0^1 f(x)dx + \epsilon \tag{13}$$

from (2) and (12). Since $\int_0^1 f(x)dx = l$ is the length of the interval J, this proves our proposition.

We shall proceed to the proof of Theorem 1. Using Lemma 1, we can choose a positive integer k such that for any integer n at least one of the k numbers $n\xi+\xi, n\xi+2\xi, \cdots, n\xi+k\xi$ is contained with modulus 1 in the interval I. From the assumption on a_n (since we can suppose $c>1$ without any loss of generality) we have

$$a_{k-m} \geqq a_k/c^m \geqq a_k/c^k \qquad (m \leqq k).$$

Then

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \left| \phi(n\xi) \right| &= \sum_{n=0}^{\infty} \sum_{m=1}^{k} a_{nk+m} \left| \phi((nk + m)\xi) \right| \\ &\geqq (d/c^k) \sum_{n=1}^{\infty} a_{nk}. \end{aligned}$$

Consequently

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} \sum_{m=1}^{k} a_{nk+m} \leqq \sum_{n=1}^{\infty} \sum_{m=1}^{k} a_{nk} c^m \\ &\leqq \frac{c^{k+1} - 1}{c - 1} \sum_{n=1}^{\infty} a_{nk} \leqq \frac{c^k(c^{k+1} - 1)}{d(c - 1)} \sum_{n=1}^{\infty} a_n \left| \phi(n\xi) \right| < \infty. \end{aligned}$$

This proves the theorem.

REMARK. This theorem implies Theorem B in §1; in fact if $\phi(x)$ is Riemann integrable, $\phi(x)$ is continuous almost everywhere, therefor the assumption $\int_0^1 |\phi(x)| dx \neq 0$ assures the existence of an interval on which $|\phi(x)|$ is greater than a positive constant.

THEOREM 2. *Besides the assumption on* $\phi(x)$ *and* $\{a_n\}$ *in Theorem* 1, *we assume that* $\phi(0)\neq 0$; *then the absolute convergence of the series* $\sum_{n=1}^{\infty} a_n\phi(nx)$ *for any real* x_0 *implies the convergence of the series* $\sum_{n=1}^{\infty} a_n$.

PROOF. If x_0 is irrational, then the result follows from Theorem 1. In case x_0 is rational, say $x_0=p/q$ where p and q are integers, the numbers nqx_0 $(n=1, 2, \cdots)$ are all equal to 0 with modulus 1. An argument quite similar to that in the proof of Theorem 1 proves also the required result.

This theorem implies Theorem A in §1 for the case of cosine series. Corresponding to the case of sine series we shall prove the following theorem.

THEOREM 3. *Let* $\phi(x)$ *and* a_n *satisfy the same conditions as in Theorem* 1. *If the series* $\sum_{n=1}^{\infty} a_n\phi(nx)$ *converges absolutely at an irrational* x_0 *or at a rational* x_0, $x_0=p/q$ *say, where* p *and* q *are relatively prime integers and there exists at least one of the numbers* r/q $(r=1, \cdots, q-1)$ *which is not the zero of* $\phi(x)$, *then the series* $\sum_{n=1}^{\infty} a_n$ *converges.*

PROOF. For irrational x_0, this reduces to Theorem 1. Let us suppose that $\phi(r_0/q)\neq 0$, $0<r_0\leqq q-1$. Since p and q are relatively prime, there are two integers u and v such that $up+vq=1$, that is, $up/q=1/q-v$. Consequently there is a number in the ur_0 consecutive numbers of the sequence $\{nx_0\}$ which is equal to r_0/q with modulus 1. Therefore the same argument as in the proof of Theorem 1 gives the required result.

3. **Double trigonometrical series.** We shall first state the following lemmas which are generalizations of the results of Mazur-Orlicz [1].

LEMMA 2. *If* $f(x)$ *is measurable, bounded, and periodic, we have for every integrable function* $g(x)$ *in* (a, b)

$$\lim_{m,n\to\infty} \int_a^b f(\omega_{mn}x+\theta_{mn})g(x)dx = \mathfrak{M}(f)\int_a^b g(x)dx,$$

where $\mathfrak{M}(f)=\int_0^l f(x)dx/l$, l *being the period of* $f(x)$, *and* $\{\omega_{mn}\}$ *and* $\{\theta_{mn}\}$ *are any sequences of real numbers such that lim* $\omega_{mn}=+\infty$.

PROOF. If $m_i, n_i\to\infty$ $(i\to\infty)$, we have by a result of Mazur and Orlicz [1, Lemma 1]

$$\lim_{i\to\infty} \int_a^b f(\omega_{m_in_i}x+\theta_{m_in_i})g(x)dx = \mathfrak{M}(f)\int_a^b g(x)dx.$$

This proves Lemma 2.

LEMMA 3. *If $f(x)$ is measurable and periodic, then we have almost everywhere,*

$$\limsup_{m,n\to\infty} |a_{mn}f(\omega_{mn}x+\theta_{mn})| = \limsup_{m,n\to\infty} |a_{mn}| \text{ ess sup } |f(x)|,$$

where $\{\omega_{mn}\}$, $\{\theta_{mn}\}$ are the same as in Lemma 2 and $\{a_{mn}\}$ is any sequence of real numbers.

PROOF. We can choose a sequence of suffixes $\{m_i, n_i\}$ such that $m_i, n_i\to\infty$ $(i\to\infty)$ and $\lim_{i\to\infty}|a_{m_in_i}| = \limsup_{m,n\to\infty}|a_{mn}|$. Then by Mazur and Orlicz [1, Theorem 1] we have almost everywhere

$$\begin{aligned}\limsup_{i\to\infty} |a_{m_in_i}f(\omega_{m_in_i}x+\theta_{m_in_i})| &= \limsup_{i\to\infty} |a_{m_in_i}| \text{ ess sup } |f(x)| \\ &= \limsup_{m,n\to\infty} |a_{mn}| \text{ ess sup } |f(x)|.\end{aligned}$$

It follows that almost everywhere

$$\limsup_{m,n\to\infty} |a_{mn}f(\omega_{mn}x+\theta_{mn})| \geqq \limsup_{m,n\to\infty} |a_{mn}| \text{ ess sup } |f(x)|.$$

Since the inverse inequality is evident, our lemma is proved.

LEMMA 4. *If $f(x)$ and $g(x)$ are linearly independent, bounded, measurable, and periodic with the same period, then for any sequences $\{a_{mn}\}$ and $\{b_{mn}\}$, we have almost everywhere*

$$\begin{aligned}&\limsup_{m,n\to\infty} |a_{mn}f(\omega_{mn}x+\theta_{mn}) + b_{mn}g(\omega_{mn}x+\theta_{mn})| \\ &\qquad\geqq c\limsup_{m,n\to\infty}(|a_{mn}|+|b_{mn}|),\end{aligned}$$

where $c=\inf_{|a|+|b|=1}\{\text{ess sup } |af(x)+bg(x)|\}>0$. (See Mazur-Orlicz [1, Theorem 3 and its corollary].)

PROOF. We can choose a sequence of indices m_i, n_i $(m_i, n_i\to\infty$ as $i\to\infty)$ such that $\limsup_{m,n\to\infty}(|a_{mn}|+|b_{mn}|) = \lim_{i\to\infty}(|a_{m_in_i}| + |b_{m_in_i}|)$. Then we have almost everywhere

$$\begin{aligned}&\limsup_{m,n\to\infty} |a_{mn}f(\omega_{mn}x+\theta_{mn}) + b_{mn}g(\omega_{mn}x+\theta_{mn})| \\ &\qquad\geqq \limsup_{i\to\infty} |a_{m_in_i}f(\omega_{m_in_i}x+\theta_{m_in_i}) + b_{m_in_i}g(\omega_{m_in_i}x+\theta_{m_in_i})| \\ &\qquad\geqq c\limsup_{i\to\infty}(|a_{m_in_i}|+|b_{m_in_i}|) \\ &\qquad= c\limsup_{m,n\to\infty}(|a_{mn}|+|b_{mn}|),\end{aligned}$$

by a result of Mazur and Orlicz [1, Theorem 3 and its corollary]. This proves the lemma.

We shall now prove the following theorem which is a generalization of Cantor-Lebesgue's theorem for double trigonometric series.

THEOREM 4. *Let $f(x)$, $g(x)$ be any measurable, periodic functions, and $\phi(y)$, $\psi(y)$ be linearly independent, bounded, measurable, and periodic functions with the same period, and let $\{\lambda_{mn}\}$, $\{\omega_{mn}\}$, and $\{\omega'_{mn}\}$ be the sequences which tend to ∞ as $m, n \to \infty$, and $\{\tau_{mn}\}$, $\{\theta_{mn}\}$, $\{\theta'_{mn}\}$ be any sequences. Then for any sequences $\{p_{mn}\}$ and $\{q_{mn}\}$, there exists a constant $c>0$ depending only on ϕ and ψ, such that*

$$\begin{aligned}\limsup_{m,n\to\infty} \big| p_{mn}f(\omega_{mn}x+\theta_{mn})\phi(\lambda_{mn}y+\tau_{mn}) \\ + q_{mn}g(\omega'_{mn}x+\theta'_{mn})\psi(\lambda_{mn}y+\tau_{mn}) \big| \\ \geqq c\max\Big\{\limsup_{m,n\to\infty} |p_{mn}| \operatorname{ess\,sup} |f(x)|, \\ \limsup_{m,n\to\infty} |q_{mn}| \operatorname{ess\,sup} |g(x)| \Big\}\end{aligned}$$

for almost every (x, y).

PROOF. If we fix an x, we have

$$\begin{aligned}\limsup_{m,n\to\infty} \big| p_{mn}f(\omega_{mn}x+\theta_{mn})\phi(\lambda_{mn}y+\tau_{mn}) \\ + q_{mn}g(\omega'_{mn}x+\theta'_{mn})\psi(\lambda_{mn}y+\tau_{mn}) \big| \\ \geqq c\limsup_{m,n\to\infty} \big\{ |p_{mn}f(\omega_{mn}x+\theta_{mn})| + |q_{mn}g(\omega'_{mn}x+\theta'_{mn})| \big\} \\ \geqq c\max\Big\{\limsup_{m,n\to\infty} |p_{mn}f(\omega_{mn}x+\theta_{mn})|, \\ \limsup_{m,n\to\infty} |q_{mn}g(\omega'_{mn}x+\theta'_{mn})| \Big\}\end{aligned}$$

by Lemma 4. Applying Lemma 3, we have for almost every x

$$\limsup_{m,n\to\infty} |p_{mn}f(\omega_{mn}x+\theta_{mn})| = \limsup_{m,n\to\infty} |p_{mn}| \operatorname{ess\,sup} |f(x)|$$

and

$$\limsup_{m,n\to\infty} |q_{mn}g(\omega'_{mn}x+\theta'_{mn})| = \limsup_{m,n\to\infty} |q_{mn}| \operatorname{ess\,sup} |g(x)|.$$

Then by Fubini's theorem we get the theorem.

COROLLARY. *Let*

$$A_{mn}(x, y) = a_{mn} \cos mx \cos ny + b_{mn} \sin mx \sin ny$$
$$+ c_{mn} \cos mx \sin ny + d_{mn} \sin mx \sin ny.$$

If $A_{mn}(x, y)$ tends to zero as $m, n \to \infty$, for every (x, y) belonging to a plane set of positive measure, then

$$\rho_{mn} = (a_{mn}^2 + b_{mn}^2 + c_{mn}^2 + d_{mn}^2)^{1/2} \to 0 \qquad \text{as } m, n \to \infty.$$

This is the Reves-Szász generalization of the Cantor-Lebesgue theorem [**1**, Theorem 1].

PROOF. Let us put in Theorem 4

$$f(x) = g(x) = \cos x, \qquad \phi(y) = \cos y, \qquad \psi(y) = \sin y,$$
$$p_{mn}^2 = a_{mn}^2 + b_{mn}^2, \qquad q_{mn}^2 = c_{mn}^2 + d_{mn}^2.$$

Then we may write (see Reves and Szász [**2**, Theorem 1])

$$A_{mn}(x, y) = p_{mn} \cos (mx - \theta_{mn}) \cos ny + q_{mn} \cos (mx - \theta'_{mn}) \sin ny.$$

Therefore we get the corollary by Theorem 4.

THEOREM 5. *Let $A_{mn}(x, y)$ and ρ_{mn} be the same as in the above corollary. Then the plane set E,*[3]

$$E = \left\{(x, y) \,\middle|\, \limsup_{m,n\to\infty} \sum_{i,j=1}^{m,n} |A_{ij}(x, y)| \Big/ \sum_{i,j=1}^{m,n} \rho_{ij} < \alpha\right\}$$

is of measure zero, provided that $\alpha < 4/2^{1/2}\pi^2$ and $\sum_{m,n=1}^{\infty} \rho_{mn} = \infty$.

PROOF. Let $f(x, y)$ be the characteristic function of the set E, then we have

$$\alpha f(x, y) \geqq \limsup_{m,n\to\infty} f(x, y) \sum_{i,j=1}^{m,n} |A_{ij}(x, y)| \Big/ \sum_{i,j=1}^{m,n} \rho_{ij}$$
$$= \limsup_{m,n\to\infty} \frac{f(x, y) \sum_{i,j=1}^{m,n} |A_{ij}(x, y)|}{\sum_{i,j=1}^{m,n} (\Delta_{ij}(x))^{1/2}} \frac{\sum_{i,j=1}^{m,n} (\Delta_{ij}(x))^{1/2}}{\sum_{i,j=1}^{m,n} \rho_{ij}}$$
$$\geqq \limsup_{m,n\to\infty} \frac{1}{2^{1/2}} \frac{f(x, y) \sum_{i,j=1}^{m,n} |A_{ij}(x, y)|}{\sum_{i,j=1}^{m,n} (\Delta_{ij}(x))^{1/2}}$$
$$\cdot \frac{\sum_{i,j=1}^{m,n} \{ | p_{ij} \cos (ix - \theta_{ij}) | + | q_{ij} \cos (ix - \theta'_{ij}) | \}}{\sum_{i,j=1}^{m,n} \rho_{ij}}$$

[3] Of course, we suppose that E is contained in the square $(0, 2\pi; 0, 2\pi)$.

where $\Delta_{ij}(x)=p_{ij}^2\cos^2(ix-\theta_{ij})+q_{ij}^2\cos^2(ix-\theta'_{ij})$.

Applying Lebesgue's theorem we get

$$\alpha\int_0^{2\pi} f(x, y)dy \geqq \frac{1}{2^{1/2}}\limsup_{m,n\to\infty}\frac{\sum_{i,j=1}^{m,n}\int_0^{2\pi} f(x, y)\,|\,A_{ij}(x, y)\,|\,dy}{\sum_{i,j=1}^{m,n}(\Delta_{ij}(x))^{1/2}}$$

$$\cdot\frac{\sum_{i,j=1}^{m,n}\{\,|\,p_{ij}\cos(ix-\theta_{ij})\,|+|\,q_{ij}\cos(ix-\theta'_{ij})\,|\,\}}{\sum_{i,j=1}^{m,n}\rho_{ij}}.$$

Since $\sum\rho_{mn}=\infty$, we can easily verify that

$$\sum_{i,j=1}^{\infty}(\Delta_{ij}(x))^{1/2}\geqq\frac{1}{2^{1/2}}\sum_{i,j=1}^{\infty}\{\,|\,p_{ij}\cos(ix-\theta_{ij})\,|$$

$$+\,|\,q_{ij}\cos(ix-\theta'_{ij})\,|\,\}=\infty$$

for almost every x. For such x, we have by Lemma 2

$$\lim_{m,n\to\infty}\frac{\sum_{i,j=1}^{m,n}\int_0^{2\pi} f(x, y)\,|\,A_{ij}(x, y)\,|\,dy}{\sum_{i,j=1}^{m,n}(\Delta_{ij}(x))^{1/2}}=\frac{2}{\pi}\int_0^{2\pi} f(x, y)dy.$$

Hence

$$\alpha\int_0^{2\pi} f(x, y)dy\geqq\frac{2}{2^{1/2}\pi}\int_0^{2\pi} f(x, y)dy$$

$$\cdot\limsup_{m,n\to\infty}\frac{\sum_{i,j=1}^{m,n}\{\,|\,p_{ij}\cos(ix-\theta_{ij})\,|+|\,q_{ij}\cos(ix-\theta'_{ij})\,|\,\}}{\sum_{i,j=1}^{m,n}\rho_{ij}}.$$

Since by Lemma 2

$$\lim_{i,j\to\infty}\int_0^{2\pi}\int_0^{2\pi} f(x, y)\,|\cos(ix-\theta_{ij})\,|\,dxdy$$

$$=\frac{2}{\pi}\int_0^{2\pi}\int_0^{2\pi} f(x, y)dxdy,$$

we have

$$\alpha\int_0^{2\pi}\int_0^{2\pi} f(x, y)dxdy\geqq\frac{4}{2^{1/2}\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(x, y)dxdy.$$

If $\alpha<4/2^{1/2}\pi^2$, then we get

$$\int_0^{2\pi}\int_0^{2\pi} f(x, y)dxdy=0,$$

that is, E is of measure zero. Thus we get the theorem.

This is a generalization of a theorem of Salem [**3**, Theorem X]. Denjoy-Lusin's theorem for double trigonometrical series, which was given by Reves and Szász [**2**, Theorem 2], can be easily derived from Theorem 4.

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