NOTES ON FOURIER ANALYSIS. XXX
ON THE ABSOLUTE CONVERGENCE OF
CERTAIN SERIES OF FUNCTIONS

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1. Introduction. Recently O. Szász [4] has proved the following
theorems which are generalizations of Fatou's theorem (cf. Zygmund
[5, p. 132]).

**Theorem A.** If the series
\[ \sum_{n=1}^{\infty} a_n \cos nx, \quad |a_{n+1}| \leq c|a_n| \quad (c>0); \quad n = 1, 2, \ldots, \]
is absolutely convergent at a point \( x_0 \), then \( \sum_{n=1}^{\infty} |a_n| < \infty \). The same
is true for the series \( \sum_{n=1}^{\infty} a_n \sin nx \), provided that \( x_0 \not\equiv 0 \) (mod \( \pi \)).

**Theorem B.** Let \( \phi(x) \) be Riemann integrable in the interval (0, 1)
and periodic with period 1 and \( \int_0^1 |\phi(x)| \, dx = 0 \). If the series
\[ \sum_{n=1}^{\infty} a_n \phi(nx), \quad 0 < |a_{n+1}| < (1+c/n) |a_n| \quad (c>0; \quad n = 1, 2, \ldots), \]
is absolutely convergent at an irrational point \( x_0 \), then \( \sum_{n=1}^{\infty} |a_n| < \infty \).

There are some gaps between the conditions for the sequences of
coefficients \( \{a_n\} \) in these two theorems. In §2, we shall prove that
the conditions for \( \{a_n\} \) in Theorem B can be replaced by those in
Theorem A.

On the other hand Reves and Szász [2] have proved the analogues
of Cantor-Lebesgue's theorem and Denjoy-Lusin's theorem for the
double trigonometric series. In §3, we shall generalize these theorems:
the former is on the line of Mazur-Orlicz' generalization for Cantor-
Lebesgue's theorem [1, Theorem 1 and its corollary] and the latter
is in the direction of Salem's generalization for Denjoy-Lusin's
theorem [3, Theorem X].

2. Theorems of Fatou-Szász type. We shall give here the gen-
eralizations of Theorems A and B.

**Theorem 1.** Let \( \phi(x) \) be defined in the interval (0, 1) and periodic
with period 1 and suppose that there exists an interval \( I = (a, b) \),

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1 The authors express their hearty thanks to the referee who gave many valuable
suggestions, especially that the original proof of Theorem 1 was incomplete.
2 Numbers in brackets refer to the literature at the end of the paper.

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\[ 0 \leq a < b \leq 1, \text{ on which } \phi(x) \text{ is greater than a positive constant } d. \text{ If the series } \sum_{n=1}^{\infty} a_n \phi(nx) (a_n \geq 0) \text{ is absolutely convergent at an irrational point } \xi \text{ and if } 0 < a_{n+1} \leq c a_n (c > 0; n = 1, 2, \cdots), \text{ then } \sum_{n=1}^{\infty} a_n < \infty. \]

For the proof we need the following lemma.

**Lemma 1.** Let \( J \) be an interval and \( \theta \) be an irrational number contained in the interval \((0, 1)\). Then for an arbitrary real number \( \alpha \) there exists a positive integer \( N \) independent of \( \alpha \) such that at least one of the numbers \( \alpha + n\theta, n = 1, 2, \cdots, N, \) is contained in the interval \( J \) with modulus 1.

**Proof.** We shall prove more precisely that if we denote the number of the points in the sequence \( \{\alpha + n\theta\}, n = 1, 2, \cdots, n, \) which are contained with modulus 1 in the interval \( J \) by \( n_J(\alpha) \), then the ratio \( n_J(\alpha)/n \) tends independently on \( \alpha \) to the length of the interval \( J \) as \( n \to \infty \); that is, for any given \( \epsilon > 0 \), there is a positive integer \( N \) independent of \( \alpha \) such that \( |n_J(\alpha)/n - l| < \epsilon \) for \( n > N \), where \( l \) is the length of the interval \( J \).

To prove this, let \( f(x) \) be the function which is obtained by extending the characteristic function of the interval \( J \) with period 1. Then there exist two trigonometrical polynomials \( p(x) \) and \( P(x) \) which satisfy the inequalities

\[
1. \quad p(x) \leq f(x) \leq P(x) \quad \text{ in } (0, 1),
\]

\[
2. \quad \int_0^1 [f(x) - p(x)] dx < \epsilon/2, \quad \int_0^1 [P(x) - f(x)] dx < \epsilon/2.
\]

We can write

\[
3. \quad p(x) = \sum_{k=-m}^{m} c_k e^{2\pi i k x}, \quad P(x) = \sum_{k=-m}^{m} C_k e^{2\pi i k x}.
\]

Remembering that

\[
4. \quad \frac{n_J(\alpha)}{n} = \frac{1}{n} \sum_{r=1}^{n} f(\alpha + r\theta),
\]

we have

\[
5. \quad \frac{1}{n} \sum_{r=1}^{n} p(\alpha + r\theta) \leq \frac{n_J(\alpha)}{n} \leq \frac{1}{n} \sum_{r=1}^{n} P(\alpha + r\theta)
\]
by (1).

On the other hand

\[
\frac{1}{n} \sum_{r=1}^{n} \phi(\alpha + r\theta) = \frac{1}{n} \sum_{r=1}^{n} \sum_{k=-m}^{m} c_k e^{2\pi ik(n+1)\theta}
\]

\[
= \frac{1}{n} \sum_{r=1}^{n} \sum_{k=-m}^{m} c_k e^{2\pi ik\theta} e^{2\pi ikn\theta}
\]

\[
= c_0 + \frac{1}{n} \sum_{k=-m}^{m} c_k e^{2\pi ik\theta} \sum_{r=1}^{n} e^{2\pi ikn\theta}
\]

\[
= c_0 + \frac{1}{n} \sum_{k=-m}^{m} c_k e^{2\pi ik\theta} \frac{e^{2\pi ink\theta} - e^{2\pi i(n+1)k\theta}}{1 - e^{2\pi i\theta}},
\]

where $\sum'$ denotes the summation omitting the term for $k = 0$. Thus we have

\[
\left| \int_0^1 \phi(x)dx - \frac{1}{n} \sum_{r=1}^{n} \phi(\alpha + r\theta) \right|
\]

\[
\leq \frac{1}{n} \left| \sum_{k=-m}^{m} c_k e^{2\pi ik\theta} \frac{e^{2\pi ink\theta} - e^{2\pi i(n+1)k\theta}}{1 - e^{2\pi i\theta}} \right|
\]

\[
\leq \frac{1}{n} \sum_{k=-m}^{m} \left| c_k \right| \frac{2}{1 - e^{2\pi i\theta}}
\]

\[
\leq \frac{2\Delta}{n} \sum_{k=-m}^{m} \left| c_k \right|
\]

where

\[
\Delta = \max_{-m \leq k \leq m, k \neq 0} \frac{1}{1 - e^{2\pi i\theta}}
\]

and $\Delta$ is finite since $\theta$ is irrational and $m$ is fixed; moreover it is evident from the definition that $\Delta$ is independent of $\alpha$.

Therefore if we choose $N_1$ such that

\[
N_1 > 4\Delta \sum_{k=-m}^{m} \left| c_k \right| / \varepsilon,
\]

we have

\[
\left| \int_0^1 \phi(x)dx - \frac{1}{n} \sum_{r=1}^{n} \phi(\alpha + r\theta) \right| < \varepsilon/2 \quad \text{for } n > N_1.
\]

Quite similarly we can choose $N_2$ such that
(11) \[ \left| \int_0^1 P(x)dx - \frac{1}{n} \sum_{r=1}^{n} P(\alpha + r\theta) \right| < \epsilon/2 \quad \text{for } n > N_2. \]

Let us take \( N = \max (N_1, N_2) \), then we have

\[ \int_0^1 P(x)dx - \epsilon/2 < \frac{1}{n} \sum_{r=1}^{n} P(\alpha + r\theta) \leq \frac{n_f(\alpha)}{n} \leq \frac{1}{n} \sum_{r=1}^{n} P(\alpha + r\theta) < \int_0^1 P(x)dx + \epsilon/2 \]

from (5), (10), and (11), and finally

\[ \int_0^1 f(x)dx - \epsilon \leq \frac{n_f(\alpha)}{n} \leq \int_0^1 f(x)dx + \epsilon \]

from (2) and (12). Since \( \int_0^1 f(x)dx = l \) is the length of the interval \( J \), this proves our proposition.

We shall proceed to the proof of Theorem 1. Using Lemma 1, we can choose a positive integer \( k \) such that for any integer \( n \) at least one of the \( k \) numbers \( n\xi + \xi, n\xi + 2\xi, \ldots, n\xi + k\xi \) is contained with modulus 1 in the interval \( I \). From the assumption on \( a_n \) (since we can suppose \( c > 1 \) without any loss of generality) we have

\[ a_{k-m} \geq a_k/c^m \geq a_k/c^k \quad (m \leq k). \]

Then

\[ \sum_{n=1}^{\infty} a_n |\phi(n\xi)| = \sum_{n=0}^{k} \sum_{m=1}^{n} a_{nk+m} |\phi((nk+m)\xi)| \]

\[ \geq (d/c^k) \sum_{n=1}^{\infty} a_{nk}. \]

Consequently

\[ \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{k} \sum_{m=1}^{n} a_{nk+m} \leq \sum_{n=1}^{k} \sum_{m=1}^{n} a_{nk}c^m \]

\[ \leq \frac{c^{k+1} - 1}{c - 1} \sum_{n=1}^{\infty} a_{nk} \leq \frac{c^k(c^{k+1} - 1)}{d(c - 1)} \sum_{n=1}^{\infty} a_n |\phi(n\xi)| < \infty. \]

This proves the theorem.

Remark. This theorem implies Theorem B in §1; in fact if \( \phi(x) \) is Riemann integrable, \( \phi(x) \) is continuous almost everywhere, therefore the assumption \( \int_0^1 |\phi(x)| dx \neq 0 \) assures the existence of an interval on which \( |\phi(x)| \) is greater than a positive constant.
Theorem 2. Besides the assumption on $\phi(x)$ and $\{a_n\}$ in Theorem 1, we assume that $\phi(0) \neq 0$; then the absolute convergence of the series $\sum_{n=1}^{\infty} a_n \phi(nx)$ for any real $x_0$ implies the convergence of the series $\sum_{n=1}^{\infty} a_n$.

Proof. If $x_0$ is irrational, then the result follows from Theorem 1. In case $x_0$ is rational, say $x_0 = p/q$ where $p$ and $q$ are integers, the numbers $nx_0$ ($n = 1, 2, \cdots$) are all equal to $0$ with modulus $1$. An argument quite similar to that in the proof of Theorem 1 proves also the required result.

This theorem implies Theorem A in §1 for the case of cosine series. Corresponding to the case of sine series we shall prove the following theorem.

Theorem 3. Let $\phi(x)$ and $a_n$ satisfy the same conditions as in Theorem 1. If the series $\sum_{n=1}^{\infty} a_n \phi(nx)$ converges absolutely at an irrational $x_0$ or at a rational $x_0$, $x_0 = p/q$ say, where $p$ and $q$ are relatively prime integers and there exists at least one of the numbers $r/q$ ($r = 1, \cdots, q-1$) which is not the zero of $\phi(x)$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

Proof. For irrational $x_0$, this reduces to Theorem 1. Let us suppose that $\phi(r_0/q) \neq 0$, $0 < r_0 \leq q - 1$. Since $p$ and $q$ are relatively prime, there are two integers $u$ and $v$ such that $up + vq = 1$, that is, $up/q = 1/q - v$. Consequently there is a number in the $wr_0$ consecutive numbers of the sequence $\{nx_0\}$ which is equal to $r_0/q$ with modulus $1$. Therefore the same argument as in the proof of Theorem 1 gives the required result.

3. Double trigonometrical series. We shall first state the following lemmas which are generalizations of the results of Mazur-Orlicz [1].

Lemma 2. If $f(x)$ is measurable, bounded, and periodic, we have for every integrable function $g(x)$ in $(a, b)$

$$
\lim_{m,n \to \infty} \int_a^b f(\omega_n x + \theta_m)g(x)dx = \mathcal{M}(f) \int_a^b g(x)dx,
$$

where $\mathcal{M}(f) = \int_a^b f(x)dx/l$, $l$ being the period of $f(x)$, and $\omega_{nm}$ and $\theta_{nm}$ are any sequences of real numbers such that $\lim \omega_{nm} = +\infty$.

Proof. If $m_i, n_i \to \infty$ ($i \to \infty$), we have by a result of Mazur and Orlicz [1, Lemma 1]

$$
\lim_{i \to \infty} \int_a^b f(\omega_{m_i} x + \theta_{m_i} n_i)g(x)dx = \mathcal{M}(f) \int_a^b g(x)dx.
$$
This proves Lemma 2.

**Lemma 3.** If \( f(x) \) is measurable and periodic, then we have almost everywhere,

\[
\lim_{m,n \to \infty} \sup |a_{mn}f(\omega_{mn}x + \theta_{mn})| = \lim_{m,n \to \infty} \sup |a_{mn}| \text{ ess sup } |f(x)|,
\]

where \( \{\omega_{mn}\}, \{\theta_{mn}\} \) are the same as in Lemma 2 and \( \{a_{mn}\} \) is any sequence of real numbers.

**Proof.** We can choose a sequence of suffixes \( m_i, n_i \) such that \( m_i, n_i \to \infty \) (\( i \to \infty \)) and \( \lim_{i \to \infty} |a_{m_i,n_i}| = \limsup_{m,n \to \infty} |a_{mn}| \). Then by Mazur and Orlicz [1, Theorem 1] we have almost everywhere

\[
\lim_{i \to \infty} |a_{m_i,n_i}f(\omega_{m_i,n_i}x + \theta_{m_i,n_i})| = \lim_{i \to \infty} |a_{m_i,n_i}| \text{ ess sup } |f(x)|
\]

\[
= \lim_{m,n \to \infty} |a_{mn}| \text{ ess sup } |f(x)|.
\]

It follows that almost everywhere

\[
\lim_{m,n \to \infty} |a_{mn}f(\omega_{mn}x + \theta_{mn})| \geq \lim_{m,n \to \infty} |a_{mn}| \text{ ess sup } |f(x)|.
\]

Since the inverse inequality is evident, our lemma is proved.

**Lemma 4.** If \( f(x) \) and \( g(x) \) are linearly independent, bounded, measurable, and periodic with the same period, then for any sequences \( \{a_{mn}\} \) and \( \{b_{mn}\} \), we have almost everywhere

\[
\lim_{m,n \to \infty} |a_{mn}f(\omega_{mn}x + \theta_{mn}) + b_{mn}g(\omega_{mn}x + \theta_{mn})| \geq c \lim_{m,n \to \infty} (|a_{mn}| + |b_{mn}|),
\]

where \( c = \inf_{|a|,|b| > 0} \{\text{ess sup } |af(x) + bg(x)| \} > 0 \). (See Mazur-Orlicz [1, Theorem 3 and its corollary].)

**Proof.** We can choose a sequence of indices \( m_i, n_i \) (\( m_i, n_i \to \infty \) as \( i \to \infty \)) such that \( \limsup_{m,n \to \infty} (|a_{mn}| + |b_{mn}|) = \lim_{i \to \infty} (|a_{m_i,n_i}| + |b_{m_i,n_i}|) \). Then we have almost everywhere

\[
\lim_{m,n \to \infty} |a_{mn}f(\omega_{mn}x + \theta_{mn}) + b_{mn}g(\omega_{mn}x + \theta_{mn})| \geq \lim_{i \to \infty} |a_{m_i,n_i}f(\omega_{m_i,n_i}x + \theta_{m_i,n_i}) + b_{m_i,n_i}g(\omega_{m_i,n_i}x + \theta_{m_i,n_i})|
\]

\[
\geq c \lim_{i \to \infty} (|a_{m_i,n_i}| + |b_{m_i,n_i}|)
\]

\[
= c \lim_{m,n \to \infty} (|a_{mn}| + |b_{mn}|),
\]
by a result of Mazur and Orlicz [1, Theorem 3 and its corollary]. This proves the lemma.

We shall now prove the following theorem which is a generalization of Cantor-Lebesgue's theorem for double trigonometric series.

**Theorem 4.** Let $f(x), g(x)$ be any measurable, periodic functions, and $\phi(y), \psi(y)$ be linearly independent, bounded, measurable, and periodic functions with the same period, and let $\lambda_{mn}, \omega_{mn}$, and $\omega'_{mn}$ be the sequences which tend to $\infty$ as $m, n \to \infty$, and $\tau_{mn}, \theta_{mn}, \theta'_{mn}$ be any sequences. Then for any sequences $\{p_{mn}\}$ and $\{q_{mn}\}$, there exists a constant $c > 0$ depending only on $\phi$ and $\psi$, such that

$$\limsup_{m,n \to \infty} \left| \sum_{m,n} p_{mn} f(\omega_{mn} x + \theta_{mn}) \phi(\lambda_{mn} y + \tau_{mn}) + q_{mn} g(\omega'_{mn} x + \theta'_{mn}) \psi(\lambda_{mn} y + \tau_{mn}) \right| \leq c \max \left\{ \limsup_{m,n \to \infty} |p_{mn}|, \limsup_{m,n \to \infty} |q_{mn}| \right\}$$

for almost every $(x, y)$.

**Proof.** If we fix an $x$, we have

$$\limsup_{m,n \to \infty} \left| \sum_{m,n} p_{mn} f(\omega_{mn} x + \theta_{mn}) \phi(\lambda_{mn} y + \tau_{mn}) + q_{mn} g(\omega'_{mn} x + \theta'_{mn}) \psi(\lambda_{mn} y + \tau_{mn}) \right| \leq c \limsup_{m,n \to \infty} \left\{ |p_{mn}| f(\omega_{mn} x + \theta_{mn}) | + |q_{mn}| g(\omega'_{mn} x + \theta'_{mn}) \right\}$$

by Lemma 4. Applying Lemma 3, we have for almost every $x$

$$\limsup_{m,n \to \infty} |p_{mn}| f(\omega_{mn} x + \theta_{mn}) | = \limsup_{m,n \to \infty} |p_{mn}| f(x)$$

and

$$\limsup_{m,n \to \infty} |q_{mn}| g(\omega'_{mn} x + \theta'_{mn}) | = \limsup_{m,n \to \infty} |q_{mn}| g(x)$$

Then by Fubini's theorem we get the theorem.

**Corollary.** Let
\[ A_{mn}(x, y) = a_{mn} \cos mx \cos ny + b_{mn} \sin mx \sin ny \]
\[ + c_{mn} \cos mx \sin ny + d_{mn} \sin mx \sin ny. \]

If \( A_{mn}(x, y) \) tends to zero as \( m, n \to \infty \), for every \((x, y)\) belonging to a plane set of positive measure, then

\[ \rho_{mn} = \left( a_{mn}^2 + b_{mn}^2 + c_{mn}^2 + d_{mn}^2 \right)^{1/2} \to 0 \quad \text{as} \quad m, n \to \infty. \]

This is the Reves-Szász generalization of the Cantor-Lebesgue theorem [1, Theorem 1].

**Proof.** Let us put in Theorem 4

\[ f(x) = g(x) = \cos x, \quad \phi(y) = \cos y, \quad \psi(y) = \sin y, \]

Then we may write (see Reves and Szász [2, Theorem 1])

\[ A_{mn}(x, y) = \rho_{mn} \cos (mx - \theta_{mn}) \cos ny + q_{mn} \cos (mx - \theta_{mn}') \sin ny. \]

Therefore we get the corollary by Theorem 4.

**Theorem 5.** Let \( A_{mn}(x, y) \) and \( \rho_{mn} \) be the same as in the above corollary. Then the plane set \( E \)

\[ E = \left\{ \left( x, y \right) \mid \limsup_{m,n \to \infty} \sum_{i,j=1}^{m,n} \left| A_{ij}(x, y) \right| \rho_{ij} < \alpha \right\} \]

is of measure zero, provided that \( \alpha < 4/2^{1/2} \pi^2 \) and \( \sum_{m,n=1}^{\infty} \rho_{mn} = \infty \).

**Proof.** Let \( f(x, y) \) be the characteristic function of the set \( E \), then we have

\[ \alpha f(x, y) \geq \limsup_{m,n \to \infty} f(x, y) \sum_{i,j=1}^{m,n} \left| A_{ij}(x, y) \right| \rho_{ij} \]

\[ = \limsup_{m,n \to \infty} \frac{f(x, y) \sum_{i,j=1}^{m,n} \left| A_{ij}(x, y) \right| \Delta_{ij}(x)^{1/2}}{\sum_{i,j=1}^{m,n} \Delta_{ij}(x)^{1/2}} \rho_{ij} \]

\[ \geq \limsup_{m,n \to \infty} \frac{1}{2^{1/2}} \frac{f(x, y) \sum_{i,j=1}^{m,n} \left| A_{ij}(x, y) \right|}{\sum_{i,j=1}^{m,n} \Delta_{ij}(x)^{1/2}} \]

\[ \sum_{i,j=1}^{m,n} \left\{ \left| \rho_{ij} \cos (ix - \theta_{ij}) \right| + \left| q_{ij} \cos (ix - \theta_{ij}') \right| \right\} \rho_{ij} \]

\[ \sum_{i,j=1}^{m,n} \rho_{ij} \]

\[ \text{Of course, we suppose that } E \text{ is contained in the square } (0, 2\pi; 0, 2\pi). \]
where $\Delta_i(x) = p_i^2 \cos^2 (ix - \theta_{ij}) + q_i^2 \cos^2 (ix - \theta_{ij})$.

Applying Lebesgue's theorem we get

$$
\alpha \int_0^{2\pi} f(x, y) dy \geq \frac{1}{2^{1/2}} \limsup_{m,n \to \infty} \frac{\sum_{i,j=1}^{m,n} \int_0^{2\pi} f(x, y) |A_{i}(x, y)| dy}{\sum_{i,j=1}^{m,n} (\Delta_i(x))^{1/2}} \sum_{i,j=1}^{m,n} \left\{ |p_{ij} \cos (ix - \theta_{ij})| + |q_{ij} \cos (ix - \theta_{ij})| \right\}
$$

Since $\sum_{m,n} = \infty$, we can easily verify that

$$
\sum_{i,j=1}^{m,n} (\Delta_i(x))^{1/2} \geq \frac{1}{2^{1/2}} \sum_{i,j=1}^{m,n} \left\{ |p_{ij} \cos (ix - \theta_{ij})| + |q_{ij} \cos (ix - \theta_{ij})| \right\} = \infty
$$

for almost every $x$. For such $x$, we have by Lemma 2

$$
\lim_{m,n \to \infty} \frac{\sum_{i,j=1}^{m,n} \int_0^{2\pi} f(x, y) |A_{i}(x, y)| dy}{\sum_{i,j=1}^{m,n} (\Delta_i(x))^{1/2}} = \frac{2}{\pi} \int_0^{2\pi} f(x, y) dy.
$$

Hence

$$
\alpha \int_0^{2\pi} f(x, y) dy \geq \frac{2}{2^{1/2}} \int_0^{2\pi} f(x, y) dy \cdot \limsup_{m,n \to \infty} \frac{\sum_{i,j=1}^{m,n} \left\{ |p_{ij} \cos (ix - \theta_{ij})| + |q_{ij} \cos (ix - \theta_{ij})| \right\}}{\sum_{i,j=1}^{m,n} p_{ij}}
$$

Since by Lemma 2

$$
\lim_{i,j \to \infty} \int_0^{2\pi} \int_0^{2\pi} f(x, y) \cos (ix - \theta_{ij}) \, dxdy = \frac{2}{\pi} \int_0^{2\pi} \int_0^{2\pi} f(x, y) \, dxdy,
$$

we have

$$
\alpha \int_0^{2\pi} \int_0^{2\pi} f(x, y) \, dxdy \geq \frac{4}{2^{1/2} \pi^2} \int_0^{2\pi} \int_0^{2\pi} f(x, y) \, dxdy.
$$

If $\alpha < 4/2^{1/2} \pi^2$, then we get

$$
\int_0^{2\pi} \int_0^{2\pi} f(x, y) \, dxdy = 0,
$$
that is, $E$ is of measure zero. Thus we get the theorem.

This is a generalization of a theorem of Salem [3, Theorem X].

Denjoy-Lusin's theorem for double trigonometrical series, which was given by Reves and Szász [2, Theorem 2], can be easily derived from Theorem 4.

**Literature**


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