REMARKS ON INCOMPLETENESS OF $\{e^{i\lambda_n x}\}$, NON-AVERAGING SETS, AND ENTIRE FUNCTIONS[1]

R. M. REDHEFFER

If $\{\lambda_n\}$ is a set of real, nonnegative, and unequal numbers, as assumed throughout this paper, then completeness of the set $\{e^{i\lambda_n x}\}$ follows from certain conditions on the density of $\{\lambda_n\}$, or on the behavior of the function $\Lambda(u)$ equal to the number of λ_n's $<u$. For example [1] the set is complete on any interval of length $2\pi d$ whenever the Pólya maximum density

$$\limsup_{y\to 1+} \limsup_{x\to\infty} \frac{\Lambda(xy) - \Lambda(x)}{xy - x} \tag{1}$$

is greater than d. In this note we establish a simple result in the opposite direction, that

I. *The set* $\{e^{\pm i\lambda_n x}\}$ *cannot be complete* L^2 *on any interval greater than* $2\pi d$ *,if*

$$\limsup_{y\to\infty} \limsup_{x\to\infty} \frac{\Lambda(x + y) - \Lambda(x)}{(x + y) - x} \leqq d. \tag{2}$$

As a consequence $\{\sin \lambda_n x, \cos \lambda_n x\}$ cannot be complete on any interval of length greater than $2\pi d$ if (2) holds.

The denominator of (2) is written $(x+y)-x$ to show the resemblance to (1). In (2) one considers the density of the λ's in a large subinterval, as the subinterval with its length held constant is moved to infinity. The intuitive meaning of (1) is similar, except that the length of the subinterval is now allowed to increase in proportion to its distance out. The gap between these two conditions leaves the theory in an unsatisfactory state which, it is hoped, will be remedied by other investigators.

The question arose in the theory of nonaveraging sets, that is, infinite sets of positive integers λ_n such that no one member of the set is the arithmetic mean of two others. Although much work has been done on the theory of such sets [3, 4], it is still not known whether they have to have zero density. In this connection we have the following result, stated for $d=0$ as the case of greatest interest.

Received by the editors January 9, 1950 and, in revised form, July 3, 1950 and July 14, 1950.

[1] Many of the techniques used here are to be found in References [1] and [2]. Numbers in brackets refer to the references at the end of the paper.

II. *A necessary and sufficient condition that every nonaveraging set* $\{\lambda_n\}$ *have zero density is that* $\{e^{i\lambda_n x}\}$ *be always incomplete on every interval.*

Sufficiency follows from known results on completeness; the motivation for the present work was to show necessity.

Proof of incompleteness. To establish I, introduce the function $F(z)=\Pi(1-z^2/\lambda_n^2)$. By changing a finite number of λ's, which is permissible [1], we can make all $\lambda_n>\epsilon$, say. (In this paper δ, ϵ, η are small positive numbers.) From

$$\log |F(x)| = \int_\epsilon^\infty \log |1 - x^2/u^2| \, d\Lambda(u)$$

one gets $\log |F| = \int_\epsilon^{x-\epsilon} + \int_{x+\epsilon}^\infty$ whenever $|x-\lambda_n|>\epsilon$ for all λ_n. Integrate each by parts; note that the integrated terms vanish at ϵ, ∞; and combine them at $x-\epsilon$ and $x+\epsilon$ to get a term $\sim 3\epsilon\Lambda(x)/x$. There remains

$$2\int_\epsilon^{x-\epsilon} \frac{\Lambda(u)}{u} \frac{x^2du}{x^2-u^2} - 2\int_{x+\epsilon}^\infty \frac{\Lambda(u)}{u} \frac{x^2du}{u^2-x^2}.$$

In the first write $u=xv$; in the second $u=x/v$. Changing the new limits 0 and $x/(x+\epsilon)$ in the second to ϵ/x and $1-\epsilon/x$ introduces an error which may be easily estimated, giving finally

$$\log |F(x)| = 2\int_{\epsilon/x}^{1-\epsilon/x} [\Lambda(vx)/v - v\Lambda(x/v)]dv/(1-v^2) + \theta, \tag{3}$$

$$|\theta| < 5\epsilon[\max \Lambda(u)/u] \quad \text{for} \quad x \gg \epsilon, \quad |x-\lambda_n| > \epsilon.$$

If $|x-\lambda_n|>\delta$ and $l(x)$ equals the number of λ's with $|x-\lambda_n|<\epsilon$, then we must add a new error term θ_1 with

$$0 \geqq \theta_1 \geqq l \log (\delta/\epsilon). \tag{4}$$

For

$$F(x) = \prod_{|x-\lambda_n|>\epsilon} \prod_{|x-\lambda_n|\leqq\epsilon} = \prod{}' \prod{}''.$$

In an obvious notation $\Lambda'(u)=\Lambda(u)$ for $u\leqq x-\epsilon$ and $\Lambda'=\Lambda-l$ for $u>x+\epsilon$. Use the result just obtained on $\prod'$, replace $\Lambda'(vx)$ by Λ, $\Lambda'(x/v)$ by $\Lambda-l$, note $\max \Lambda'(u)/u \leqq \max \Lambda(u)/u$ to get (3) plus a new term θ_2, $\theta_2=2\int_{\epsilon/x}^{1-\epsilon/x} lvdv/(1-v^2)$. Finally, for $\delta\leqq|x-\lambda_n|\leqq\epsilon$ as in $\prod''$ we have

$$(2\delta/x)^l \leqq \prod{}'' \leqq (2\epsilon/x)^l,$$

within a factor $[1+O(\epsilon/x)]^l$. Using these estimates with $\log F = \log \prod' + \log \prod''$ gives the result (4).

To complete the proof of I choose a y in equation (2) for which $\limsup_{x\to\infty}[\Lambda(x+y)-\Lambda(x)]/y < d+\eta/2$, then choose N so large that $\Lambda(x+y)-\Lambda(x) < (d+\eta)y$ for this y, whenever $x \geqq N$. If D is the maximum density in any interval $[ky, (k+1)y]$ beyond $N(k=0, 1, 2, \cdots)$, then we have $D \leqq d+\eta$. Add enough λ's so that in the new set $\{\lambda_n'\}$ the density for each interval beyond N is exactly D. Then $\Lambda'(x) = Dx+h(x)$ with h bounded, $|h| < A$, say. The above result (3), (4) applied to $F(z) = \prod(1-z^2/\lambda_n'^2)$ gives $\log |F(x)| \leqq (m-1) \log x$ for some integer m. The function $G(x) = F(x)(\sin \eta x/m)^m/x^m$ is $O(1/x)$, hence is an element of L^2; also $\log |G(iy)|/y \sim m(\eta/m)+\pi D$. Now we use the theorem of Paley-Wiener [2; 5], which states that the following classes of entire functions are equivalent:

(a) Those which belong to L^2 on the real axis and are $o(e^{a|z|})$;

(b) Those which can be written as $\int_{-a}^{a} f(u)e^{izu}du$ for some $f(u) \in L^2$ on $[-a, a]$.

The function G may be represented, then, as $\int_{-\pi D-2\eta}^{\pi D+2\eta} g(t)e^{ixt}dt$, with g an element of L^2. Since $G(\pm\lambda_n)=0$ and $D \leqq d+\eta$, the result I follows.

Relation to nonaveraging sets. To deduce II from I we shall need the following simple lemma concerning sets of integers. Since only slight use is made of the nonaveraging character in the proof, the result is given in general form:[2]

III. *Let* P *be a property which applies to sets of unequal integers and satisfies the following conditions*:

(a) *If the set* $\{a_1, a_2, \cdots, a_n\}$ *has the property* P, *then so does* $\{a_i-k, a_{i+1}-k, \cdots, a_j-k\}$ *for any integers* i, j, k, $1 \leqq i \leqq j \leqq n$.

(b) *There exist arbitrarily large sets with the property* P *and with density greater than or equal to* d.

Then there is an infinite set $\{b_k\}$ *for which every subset* $\{b_n, b_{n+1}, \cdots, b_m\}$ *has the property* P, *and for which the upper density is greater than or equal to* d.

By upper density is meant $\limsup \Lambda(n)/n$. It seems difficult to determine whether this may be replaced by $\lim \Lambda(n)/n$ in the theorem.

The idea of the proof is to start with a set s_1 of density greater than or equal to d, then imbed this in a larger set of density greater than or equal to d, and so on. We thus obtain a set extending to infinity in one or both directions from the initial set s_1. By fixing our attention on the successive end points of the intervals used in the con-

[2] This discussion (June 1950) is a revision of the original one.

struction, we find that the upper density is greater than or equal to d in at least one direction; and hence, allowing negative b's, the result is established.

To prove III, let S_n be the set of all sets on $[1, 2^n]$ which have property P and which have density greater than or equal to d $(n=1, 2, 3, \cdots)$. It is easy to show that no S_n is vacuous, even though the lengths of their members have a special form. Each member of S_n may be thought of as having two parts $[1, 2^{n-1}]$, $[(2^{n-1}+1), 2^n]$ (a diagram is helpful). One or the other part has density greater than or equal to d, hence belongs to S_{n-1}. Thus if s_n is an element of S_n, then there are always members $s_{n-1}, s_{n-2}, \cdots$ of S_{n-1}, $S_{n-2}, \cdots$ such that $s_n \supset s_{n-1} \supset s_{n-2} \cdots$. Construct such "nested sets" for $n=1, 2, 3, \cdots$. At least one member s_1 of S_1 belongs to infinitely many nested sets; similarly for $s_2 \supset s_1$ in the sets selected by considering s_1; and so on. Thus a diagonal process gives a single infinite nested set, whence the required set is obtained.

To establish II, suppose given an infinite nonaveraging set $\{\lambda_n\}$ which does not have zero density. Then $\limsup \Lambda(n)/n = d > 0$, and hence $\{e^{i\lambda_n x}\}$ is complete [1] on an interval greater than or equal to $2\pi d$. Conversely, suppose that the set $\{e^{i\lambda_n x}\}$ is complete on an interval equal to $2\pi d > 0$. Then the particular form of I shows that there exist arbitrarily large subsets of the λ's with density greater than or equal to d, hence by III there is an infinite nonaveraging set with upper density greater than or equal to d.

Entire functions. The result (3), (4) is a theorem concerning the entire function $F(z)$, which leads to a simple proof of Levinson's density theorem for the case of real λ_n's. First, if $\lim n/\lambda_n = d$ and $\lambda_{n+1} - \lambda_n \geqq c > 0$, then $F(x) = O(e^{\delta x})$ always for any $\delta > 0$; and $1/F(x) = O(e^{\delta x})$ whenever $|x - \lambda_n| > \epsilon$. To see this, let $h(u) = \Lambda(u)/u - d$, which approaches 0 as $u \to \infty$. We have

$$(5) \qquad [\log |F(x)|]/x \leqq \theta/x + 2\int_{\epsilon/x}^{1-\epsilon/x} p(v, x)dv/(1 - v^2)$$

where

$$(6) \qquad p(v, x) = h(vx) - h(x/v).$$

The integral in (5) equals $\int_{\epsilon/x}^{\delta} + \int_{\delta}^{1-\delta} + \int_{1-\delta}^{1-\epsilon/x}$. The first integral is less than $2\delta \max |h(u)|/(1-\delta^2) \to 0$ as $\delta \to 0$. The second is less than $\max p/\delta$, which approaches 0 as $x \to \infty$, since vx and x/v both approach ∞ with x when $v \geqq \delta$. For the third, $\lambda_{n+1} - \lambda_n \geqq c$ allows Λ to increase by at most 1, or $h(u)$ by $1/(u-1)$, when u changes an

amount c. Thus, since $p=0$ at $v=1-\epsilon/x$, p cannot exceed $(k+k)/((1-\delta)x-1)\cong 2k/x$ in $\int_{1-(k-1)c/x}^{1-kc/x}$. Hence the third integral, $\int_{1-\delta}^{1-\epsilon/x}$, is less than or equal to $\sum_k \leqq \sum (c/x)(2k/x)/(1-[1-kc/x]^2)$ $<(\delta x/c)(c/x)(2k/x)(x/kc)=2\delta/c\to 0$. The proof for $1/F$ follows easily from this.

Levinson's density theorem for real λ_n is:

IV. *If* $\lim n/\lambda_n=d$, *and* $\lambda_{n+1}-\lambda_n\geqq c>0$, *then* $|F(re^{i\theta})|^{\pm 1}=O(e^{(\pm\pi d|\sin\theta|+\epsilon)r})$ *for every* $\epsilon>0$.

To establish this easily, use the Phragmén-Lindelöf theorem on the function $G(z)=F(z)e^{i\pi(d+\epsilon)z}e^{-\epsilon z}$. It is seen that $G=O(1)$ on the positive real and imaginary axes, hence in the first quadrant; and this gives the desired result. It holds for other quadrants by symmetry. For $1/F(z)$ consider $G(z)=e^{-i\pi(d-\epsilon)z}e^{-\epsilon z}/F(z)$ with semicircular indentations around the poles. Since $|1-z^2/\lambda_n^2|\geqq|1-x^2/\lambda_n^2|$ for $z=x+iy$, re-examination of the proof for $1/F(x)$ gives $1/F(z)=O(e^{\epsilon|z|})$ on the contour consisting of the real axis and these circles. Now proceed as for $F(z)$.

Another theorem of Levinson for the special case of real λ's is contained in (3), namely,

V. *If* $\lim n/\lambda_n=d$ *and* $\lambda_{n+1}-\lambda_n\geqq c>0$, *then* $1/F'(\lambda_n)=O(e^{\delta\lambda_n})$ *for every* $\delta>0$.

To see this, use $1/F'(\lambda_n)=\lim_{x\to\lambda_n}(x-\lambda_n)[1/F(x)]$, cancel $(x-\lambda_n)$, and note that $|x-\lambda_n|<c/2$ makes $|x-\lambda_k|>c/2$ for $k\neq n$.

References

1. N. Levinson, *Gap and density theorems*, Amer. Math. Soc. Colloquium Publications, vol. 26, New York, 1940.

2. R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain*, Amer. Math. Soc. Colloquium Publications, vol. 19, New York, 1934.

3. F. A. Behrend, *On sets of integers which contain no three terms in arithmetical progression*, Proc. Nat. Acad. Sci. U.S.A. vol. 32 (1946) pp. 331–332.

4. R. Salem and D. C. Spencer, *On sets of integers which contain no three terms in arithmetical progression*, Proc. Nat. Acad. Sci. U.S.A. vol. 28 (1942) pp. 561–563.

5. M. Plancherel and G. Pólya, *Fonctions entières et integrales de Fourier multiples*, Comment. Math. Helv. vol. 9 (1936–1937) pp. 224–248.

Harvard University