ON THE BEHAVIOR OF FOURIER SINE TRANSFORMS NEAR THE ORIGIN

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1. On the open half-line $x > 0$, let $f(x)$ be a non-negative, monotone function which tends to 0 as $x \to \infty$ and behaves, as $x \to 0$, in such a way that

$$\int_{+0}^{1} xf(x) dx < \infty$$

(so that $f(x)$ need not be bounded). It is well known that the improper integral

$$F(t) = \int_{0}^{\infty} f(x) \sin tx \, dx$$

must then converge, and that

$$F(t) > 0, \quad 0 < t < \infty,$$

except when $f(x) \equiv 0$. In fact, if $t > 0$ is fixed, the integral (2) can be written in the form $a_0 - a_1 + a_2 - \cdots$, where either $a_0 = a_1 = \cdots = 0$ or

$$a_n \geq a_{n+1} \to 0, \quad \text{hence} \quad \sum_{n=0}^{\infty} (-1)^n a_n > 0.$$

It seems to be worth observing that, for small $t$, the assertion of (3) can be refined substantially, since the above assumptions on $f(x)$ imply that

$$\liminf_{t \to 0} F(t)/t > 0.$$

What is more, $F(t)/t$ must tend to a positive limit ($\leq \infty$) and the latter can be represented as

$$\lim_{t \to 0} F(t)/t = \int_{0}^{\infty} xf(x) dx,$$

where it is understood that the integral (6) can have the value $\infty$.

2. This result has an interesting implication for the Fourier-Stieltjes transforms of certain distribution functions. If, in addition

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to the above properties, \( f(x) \) satisfies
\[
\int_{0}^{\infty} f(x) \, dx = 1,
\]
let \( \delta(x) = 0 \) or \( f(x) \) according as \( x \leq 0 \) or \( x > 0 \); so that
\[
\sigma(x) = \int_{-\infty}^{x} \delta(s) \, ds, \quad \text{where} \quad -\infty < x < \infty,
\]
is a distribution function. It follows that the Fourier-Stieltjes transform of \( \sigma(x) \),
\[
\Gamma(t) = \int_{0}^{+\infty} e^{itx} \, d\sigma(x),
\]
has a derivative at \( t = 0 \) if and only if \( \sigma(x) \) has a finite first moment,
\[
\int_{0}^{\infty} x \, d\sigma(x) < \infty.
\]
On the other hand, it is known that the existence of this first moment is sufficient, but not necessary, in order that
\[
\int_{0}^{\infty} \cos tx \, d\sigma(x),
\]
the real part of \( \Gamma(t) \), be differentiable at \( t = 0 \) (A. Wintner, *The Fourier transforms of probability distributions*, Edwards Brothers, 1947, p. 19; in the example given there, \( \delta(x) \) is not monotone for small positive \( x \); however, the example is easily altered so as to comply with this condition).

3. **Proof of (6).** The "alternating" character of the improper integral (2) (cf. (4)) implies that
\[
F(t) \leq \int_{0}^{\pi/t} f(x) \sin tx \, dx.
\]
Hence
\[
F(t)/t \leq \int_{0}^{\pi/t} xf(x)(\sin tx/tx) \, dx \leq \int_{0}^{\pi/t} xf(x) \, dx,
\]
since \( \sin tx \leq tx \). Consequently, as \( t \to +0 \),
\[
\lim \sup F(t)/t \leq \int_{0}^{\infty} xf(x) \, dx.
\]
It remains to show that, as \( t \to +0 \),

\[
\lim \inf F(t)/t \geq \int_0^\infty xf(x)dx.
\]

To this end, it will first be shown that if \( X > 0 \) is arbitrary, then, as \( t \to +0 \),

\[
\lim \inf F(t)/t \geq \int_0^X xf(x)dx - \frac{1}{2} X^2 f(X).
\]

On \( 0 < x < \infty \), define three functions \( f_1(x), f_2(x), f_3(x) \) by placing them respectively equal to \( f(x), f(X), f(X) \) or to 0, 0, \( f(x) \) according as \( 0 < x \leq X \) or \( X < x < \infty \). Then \( f_1, f_2, f_3 \) satisfy the same conditions as does \( f \), so that their respective sine transforms \( F_1, F_2, F_3 \) are non-negative (for \( t > 0 \)). Since \( f = f_1 - f_2 + f_3 \), it follows that \( F = F_1 - F_2 + F_3 \). Hence, as \( t \to +0 \),

\[
\lim \inf F(t)/t \geq \lim \inf F_1(t)/t - \lim F_2(t)/t.
\]

The existence of the limits on the right-hand side is clear; in fact, since the sine transforms

\[
F_i(t) = \int_0^X f_i(x) \sin xt \, dt, \quad \text{where } i = 1, 2,
\]

are integrals over a finite interval, it follows that

\[
F_i(t)/t \to \int_0^X xf_i(x)dx, \quad \text{as } t \to +0.
\]

The inequality (9) follows from (10), (11) and the definitions of \( f_1, f_2 \).

If \( 0 < \alpha < 1 \), the monotony of \( f(x) \) shows that

\[
\int_0^X xf(x)dx \geq f(X) \int_0^X xdx = \frac{1}{2} X^2(1 - \alpha^2)f(X).
\]

Hence the inequality (9) implies that

\[
\lim \inf F(t)/t \geq \int_0^X xf(x)dx - \frac{1}{2} \alpha^2 X^2 f(X).
\]

Choose \( \alpha = \alpha(X) \) as a function of \( X \), defined in such a way that \( 0 < \alpha < 1 \), and that \( \alpha X \to \infty \) but \( \alpha^2 X^2 f(X) \to 0 \), as \( X \to \infty \). The existence of such functions \( \alpha = \alpha(X) \) is clear, since \( f(X) \to 0 \) as \( X \to \infty \). Obviously, (8) follows by letting \( X \to \infty \) in (12). This completes the proof of (6).