ON THE BEHAVIOR OF FOURIER SINE TRANSFORMS NEAR THE ORIGIN

PHILIP HARTMAN AND AUREL WINTNER

1. On the open half-line $x>0$, let $f(x)$ be a non-negative, monotone function which tends to 0 as $x\rightarrow\infty$ and behaves, as $x\rightarrow 0$, in such a way that

$$\int_{+0}^{1} xf(x)dx < \infty \tag{1}$$

(so that $f(x)$ need not be bounded). It is well known that the improper integral

$$F(t) = \int_{0}^{\infty} f(x) \sin tx\, dx \tag{2}$$

must then converge, and that

$$F(t) > 0, \quad \text{where} \quad 0 < t < \infty, \tag{3}$$

except when $f(x)\equiv 0$. In fact, if $t>0$ is fixed, the integral (2) can be written in the form $a_0-a_1+a_2-\cdots$, where either $a_0=a_1=\cdots=0$ or

$$a_n \geqq a_{n+1} \rightarrow 0, \quad \text{hence} \quad \sum_{n=0}^{\infty}(-1)^n a_n > 0. \tag{4}$$

It seems to be worth observing that, for small t, the assertion of (3) can be refined substantially, since the above assumptions on $f(x)$ imply that

$$\liminf_{t\rightarrow 0} F(t)/t > 0. \tag{5}$$

What is more, $F(t)/t$ must tend to a positive limit ($\leqq\infty$) and the latter can be represented as

$$\lim_{t\rightarrow 0} F(t)/t = \int_{0}^{\infty} xf(x)dx, \tag{6}$$

where it is understood that the integral (6) can have the value ∞.

2. This result has an interesting implication for the Fourier-Stieltjes transforms of certain distribution functions. If, in addition

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to the above properties, $f(x)$ satisfies

$$\int_0^\infty f(x)dx = 1,$$

let $\delta(x)=0$ or $f(x)$ according as $x\leqq 0$ or $x>0$; so that

$$\sigma(x) = \int_{-\infty}^{x} \delta(s)ds, \quad \text{where} \quad -\infty < x < \infty,$$

is a distribution function. It follows that the Fourier-Stieltjes transform of $\sigma(x)$,

$$\Gamma(t) = \int_0^{+\infty} e^{itx}d\sigma(x),$$

has a derivative at $t=0$ if and only if $\sigma(x)$ has a finite first moment,

$$\int_0^\infty x\,d\sigma(x) < \infty.$$

On the other hand, it is known that the existence of this first moment is sufficient, but not necessary, in order that

$$\int_0^\infty \cos tx\,d\sigma(x),$$

the real part of $\Gamma(t)$, be differentiable at $t=0$ (A. Wintner, *The Fourier transforms of probability distributions*, Edwards Brothers, 1947, p. 19; in the example given there, $\delta(x)$ is not monotone for small positive x; however, the example is easily altered so as to comply with this condition).

3. **Proof of (6).** The "alternating" character of the improper integral (2) (cf. (4)) implies that

$$F(t) \leqq \int_0^{\pi/t} f(x)\sin tx\,dx.$$

Hence

$$F(t)/t \leqq \int_0^{\pi/t} xf(x)(\sin tx/tx)dx \leqq \int_0^{\pi/t} xf(x)dx,$$

since $\sin tx \leqq tx$. Consequently, as $t\rightarrow +0$,

$$\limsup F(t)/t \leqq \int_0^\infty xf(x)dx. \tag{7}$$

It remains to show that, as $t \to +0$,

$$\liminf F(t)/t \geqq \int_0^\infty xf(x)dx. \tag{8}$$

To this end, it will first be shown that if $X>0$ is arbitrary, then, as $t \to +0$,

$$\liminf F(t)/t \geqq \int_0^X xf(x)dx - \frac{1}{2} X^2 f(X). \tag{9}$$

On $0<x<\infty$, define three functions $f_1(x), f_2(x), f_3(x)$ by placing them respectively equal to $f(x)$, $f(X)$, $f(X)$ or to 0, 0, $f(x)$ according as $0<x\leqq X$ or $X<x<\infty$. Then f_1, f_2, f_3 satisfy the same conditions as does f, so that their respective sine transforms F_1, F_2, F_3 are non-negative (for $t>0$). Since $f=f_1-f_2+f_3$, it follows that $F=F_1-F_2+F_3$. Hence, as $t \to +0$,

$$\liminf F(t)/t \geqq \lim F_1(t)/t - \lim F_2(t)/t. \tag{10}$$

The existence of the limits on the right-hand side is clear; in fact, since the sine transforms

$$F_i(t) = \int_0^X f_i(x) \sin xt\, dt, \qquad \text{where } i = 1, 2,$$

are integrals over a finite interval, it follows that

$$F_i(t)/t \to \int_0^X xf_i(x)dx, \qquad \text{as } t \to +0. \tag{11}$$

The inequality (9) follows from (10), (11) and the definitions of f_1, f_2.

If $0<\alpha<1$, the monotony of $f(x)$ shows that

$$\int_{\alpha X}^X xf(x)dx \geqq f(X) \int_{\alpha X}^X xdx = \frac{1}{2} X^2(1-\alpha^2)f(X).$$

Hence the inequality (9) implies that

$$\liminf F(t)/t \geqq \int_0^{\alpha X} xf(x)dx - \frac{1}{2}\alpha^2 X^2 f(X). \tag{12}$$

Choose $\alpha=\alpha(X)$ as a function of X, defined in such a way that $0<\alpha<1$, and that $\alpha X \to \infty$ but $\alpha^2 X^2 f(X) \to 0$, as $X \to \infty$. The existence of such functions $\alpha=\alpha(X)$ is clear, since $f(X)\to 0$ as $X\to\infty$. Obviously, (8) follows by letting $X\to\infty$ in (12). This completes the proof of (6).

THE JOHNS HOPKINS UNIVERSITY