FUNCTIONS WITH PRESCRIBED LIPSCHITZ CONDITION

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1. **Introduction.** In this note we construct a class of functions each of which satisfies at every point a Lipschitz condition of prescribed order α $(0<\alpha<1)$. The construction is based on a method given by Knopp [1][1] for construction of continuous nondifferentiable functions, and follows more particularly a construction of van der Waerden [2].

2. **The construction.** We first define a fundamental periodic function $g(t, h)$. The function $g(t, h)$ has period $2h$ in t, equals zero for even multiples of h, equals one for odd multiples of h, and is linear between successive multiples of h. It is thus a saw-tooth function of t.

Now let a number α be given, where $0<\alpha<1$. Let A be an integer for which

$$2^{2A(1-\alpha)} > 2.$$

The function we are constructing is

$$g(t) = \sum_{n=1}^{\infty} 2^{-2A\alpha n} g(t, 2^{-2An}).$$

It will be shown that $g(t)$ satisfies for each value of t a Lipschitz condition of order precisely α.

3. **Proof of Lipschitz condition.**

THEOREM. *The function $g(t)$ defined above satisfies for each value of t a Lipschitz condition of order precisely α. That is, there exist two positive constants K_1 and K_2 such that*

(a) *for any t and any Δt,*

$$|\Delta g| < K_1 |\Delta t|^{\alpha}$$

where $\Delta g = g(t+\Delta t) - g(t)$, and

(b) *for any t and for infinitely many, arbitrarily small Δt,*

$$|\Delta g| > K_2 |\Delta t|^{\alpha}.$$

PROOF. We shall denote the kth summand of the series for $g(t)$ by g_k. To prove assertion (a) we let m be the integer such that

$$2^{-2A(m+1)} < \Delta t \leqq 2^{-2Am}.$$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

Since the slope of the linear portions of $g_k(t)$ is $\pm 2^{2Ak(1-\alpha)}$,

$$|\Delta g_k| \leqq 2^{2Ak(1-\alpha)} |\Delta t| \leqq 2^{2Ak(1-\alpha)-2Am} \qquad (k \leqq m).$$

Since the maximum oscillation of $g_k(t)$ is $2^{-2A\alpha k}$,

$$|\Delta g_k| \leqq 2^{-2A\alpha k} \qquad (k > m).$$

Combining the above,

$$|\Delta g| \leqq \sum_{k=1}^{m} 2^{-2Am} 2^{2A(1-\alpha)k} + \sum_{k=m+1}^{\infty} 2^{-2A\alpha k}$$

$$\leqq \frac{2^{2A(1-\alpha)-2A\alpha m}}{2^{2A(1-\alpha)}-1} + \frac{2^{-2A\alpha(m+1)}}{1-2^{-2A\alpha}}.$$

Now $|\Delta t|^{\alpha} > 2^{-2A\alpha(m+1)}$. Therefore

$$\frac{|\Delta g|}{|\Delta t|^{\alpha}} < \frac{2^{2A}}{2^{2A(1-\alpha)}-1} + \frac{1}{1-2^{-2A\alpha}},$$

which proves assertion (a).

To prove assertion (b) we use a lemma on geometrical progressions.

LEMMA. *If the ratio in a geometric progression is positive and less than* $1/2$, *and the first term is* 1, *then the first term exceeds the sum of the remaining terms by at least*

$$\frac{1-2r}{1-r}.$$

PROOF OF LEMMA. If the progression is infinite, the sum of the terms following the first is $r/(1-r)$, and if the progression is finite the sum of the terms following the first is less than this. If $r<1/2$, $r/(1-r)<1$, so that the first term, 1, exceeds the sum of the remaining terms by at least

$$1 - r/(1-r) = (1-2r)/(1-r).$$

Turning now to the proof of (b), for a given t let Δt be equal numerically to

$$2^{-2A(m+1)}$$

and in a direction so as not to include a multiple of 2^{-2Am}, where m is any positive integer. For this Δt,

$$\Delta g_k = 0 \qquad (k > m)$$

while

$$\Delta g_k = \pm 2^{2Ak(1-\alpha)}2^{-2A(m+1)} \qquad \text{if } k \leqq m.$$

Thus

$$\begin{aligned}\Delta g &= 2^{-2A(m+1)}[\pm 2^{2Am(\alpha-1)} \pm 2^{2A(m-1)(1-\alpha)} \pm \cdots \pm 2^{2A(1-\alpha)}] \\ &= 2^{-2A-2Am\alpha}[\pm 1 \pm 2^{-2A(1-\alpha)} \pm 2^{-4A(1-\alpha)} \pm \cdots].\end{aligned}$$

Now since A was chosen so that $2^{-2A(1-\alpha)}<1/2$, we may apply the lemma and conclude that

$$|\Delta g| > 2^{-2A-2Am\alpha}\frac{1-2^{-2A(1-\alpha)}\cdot 2}{1-2^{-2A(1-\alpha)}}.$$

Now $|\Delta t|^\alpha = 2^{-2Am\alpha-2A\alpha}$. Therefore

$$\frac{|\Delta g|}{|\Delta t|^\alpha} > 2^{-2A(1-\alpha)}\frac{1-2\cdot 2^{-2A(1-\alpha)}}{1-2^{-2A(1-\alpha)}},$$

and since this is true for any m, assertion (b) is proved.

It should be remarked that for any α in the range $0<\alpha<1$ the function $g(t)$ is continuous and nondifferentiable.

Bibliography

1. K. Knopp, *Ein einfaches Verfahren zur Bildung stetiger nirgends differenzierbar Funktionen*, Math. Zeit. vol. 2 (1918) pp. 1–26.

2. B. L. van der Waerdan, *Ein einfaches Beispiel einer nicht-differenzierbaren stetigen Funktion*, Math. Zeit. vol. 32 (1930) pp. 474–475.

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