NOTE ON THE HURWITZ ZETA-FUNCTION

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It is well known that Riemann gave two proofs of the functional equation for \( \zeta(s) \), the first depending on a contour integration, the second on the transformation equation for \( \theta_3(0\mid \tau) \). Hurwitz introduced his generalized zeta-function, defined for \( R(s) > 1 \) and \( 0 < a < 1 \) by

\[
(1) \quad \zeta(s, a) = \sum_{n=0}^{\infty} (n + a)^{-s},
\]

showed that it can be continued to the entire \( s \)-plane with the exception of a simple pole at \( s = 1 \), and proved that for \( R(s) > 1 \),

\[
(2) \quad \zeta(1 - s, a) = \frac{2\Gamma(s)}{(2\pi)^s} \left\{ \cos \frac{\pi s}{2} \sum_{n=1}^{\infty} \frac{\cos 2\pi na}{n^s} \right\} + \sin \frac{\pi s}{2} \sum_{n=1}^{\infty} \frac{\sin 2\pi na}{n^s}.
\]

His method of proof depends on a contour integral and parallels Riemann's first proof. It appears to have been overlooked that the second method of Riemann can be generalized to obtain the same results. The purpose of this paper is to supply such a proof.

For \( 0 < a < 1 \) and \( x > 0 \), define

\[
(3) \quad f(a, x) = \theta_3(\pi a, ix)
\]

\[
= 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 x} \cos 2\pi na.
\]

By the transformation equation for the \( \theta \)-function,
From (3) and (4) it is easy to see that $f(a, x)$ tends to the values 1 and 0 exponentially as $x$ tends to $\infty$ and 0, respectively. Hence the two functions

\begin{align*}
F(a, s) &= \int_0^1 f(a, x)x^{s/2-1}dx, \\
G(a, s) &= \int_1^\infty (f(a, x) - 1)x^{s/2-1}dx
\end{align*}

are entire in $s$. We define

\begin{equation}
H(a, s) = F(a, s) + G(a, s) - 2/s.
\end{equation}

For a later purpose, we observe that

\begin{equation}
\frac{\partial}{\partial a} H(a, s) = \int_0^\infty \frac{\partial}{\partial a} f(a, x)x^{s/2-1}dx
\end{equation}

is also an entire function of $s$.

Now for $R(s) > 1$,

\begin{equation}
H(a, s) = \int_0^\infty (f(a, x) - 1)x^{s/2-1}dx = 2\sum_{n=1}^\infty \cos 2\pi na \int_0^\infty e^{-\pi^2 x}x^{s/2-1}dx,
\end{equation}

\begin{equation}
H(a, s) = 2\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\sum_{n=1}^\infty \frac{\cos 2\pi na}{n^s} \quad (R(s) > 1).
\end{equation}

For $R(s) < 0$,

\begin{equation*}
G(a, s) = \frac{2}{s} + \int_1^\infty f(a, x)x^{s/2-1}dx,
\end{equation*}

so that

\begin{equation*}
H(a, s) = \int_0^\infty f(a, x)x^{s/2-1}dx = \int_0^\infty f(a, x^{-1})x^{-s/2-1}dx.
\end{equation*}

By (4), therefore,
\[ H(a, s) = \sum_{n=0}^{\infty} \int_0^{\infty} e^{-\pi x(n+a)x^{-(1-s)/2-1}} dx. \]

(10) \[ H(a, s) = \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \sum_{n=0}^{\infty} \frac{1}{n+a} \quad (R(s) < 0). \]

With \( s \) replaced by \( 1-s \), (10) becomes

(11) \[ H(a, 1-s) = \pi^{-(1-s)/2} \Gamma\left(\frac{s}{2}\right) \left\{ \sum_{n=0}^{\infty} (n+a)^{-s} + \sum_{n=0}^{\infty} (n+1-a)^{-s} \right\} \]
\[ = \pi^{-(1-s)/2} \Gamma\left(\frac{s}{2}\right) \left\{ \xi(s, a) + \xi(s, 1-a) \right\} \quad (R(s) > 1). \]

Now it is easy to see, from (1), that

\[ \frac{\partial}{\partial a} \xi(s, a) = -s\xi(s, 1, a). \]

Hence, differentiating (11) and replacing \( s \) by \( s-1 \), we have

(12) \[ \frac{\partial}{\partial a} H(a, 2-s) = -(s-1)\pi^{-(s-1)/2} \Gamma\left(\frac{s-1}{2}\right) \left\{ \xi(s, a) - \xi(s, 1-a) \right\} \quad (R(s) > 2). \]

Combining (11) and (12) yields

(13) \[ 2\xi(s, a) = \frac{\pi^{s/2}}{\Gamma(s/2)} H(a, 1-s) \]
\[ - \frac{\pi^{(s-1)/2}}{(s-1)\Gamma((s-1)/2)} \frac{\partial}{\partial a} H(a, 2-s) \quad (R(s) > 2). \]

Equation (13) provides the analytic continuation of \( \xi(s, a) \), and shows that \( \xi(s, a) - (s-1)^{-1} \) is an entire function of \( s \). Replace \( s \) by \( 1-s \) in (13):

(14) \[ 2\xi(1-s, a) = \frac{\pi^{(1-s)/2}}{\Gamma((1-s)/2)} H(a, s) \]
\[ + \frac{\pi^{s/2}}{s\Gamma(-s/2)} \frac{\partial}{\partial a} H(a, 1+s). \]

For \( R(s) > 1 \), we may use (9) to get

(15) \[ \frac{\partial}{\partial a} H(a, 1+s) = -4\pi^{(1-s)/2} \Gamma\left(\frac{1+s}{2}\right) \sum_{n=1}^{\infty} \frac{\sin 2\pi na}{n^s}. \]
Substituting from (9) and (15) into (14), and simplifying by well-known formulas for the T-function, we obtain the desired relation (2).

The proof just presented does not cover the classical Riemann zeta-function, defined, for $R(s) > 1$, by

$$
\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.
$$

But it is easy to see that, for $R(s) > 1$,

$$
\zeta\left(s, \frac{1}{2}\right) = \sum_{n=0}^{\infty} \left(\frac{2n+1}{2}\right)^{-s} = 2^s \sum_{n \text{ odd}} n^{-s}
$$

(16)

$$
= 2^s \left\{ \sum_{n=1}^{\infty} n^{-s} - \sum_{n=1}^{\infty} (2n)^{-s} \right\}
$$

$$
= (2^s - 1)\zeta(s).
$$

This provides the continuation of $\zeta(s)$ and shows that it has a simple pole at $s = 1$, with possible simple poles at $s = 2n\pi i/\log 2$ ($n = 0, \pm 1, \pm 2, \cdots$). Now if we set $a = 1/2$ in equation (2), we obtain, for $R(s) > 1$,

$$
\zeta\left(1 - s, \frac{1}{2}\right) \frac{(2\pi)^s}{2\Gamma(s) \cos (\pi s/2)} = \sum_{n=1}^{\infty} (-1)^n n^{-s} = \sum_{n \text{ even}} n^{-s} - \sum_{n \text{ odd}} n^{-s}
$$

$$
= 2 \sum_{n=1}^{\infty} (2n)^{-s} - \sum_{n=1}^{\infty} n^{-s}
$$

$$
= (2^{1-s} - 1)\zeta(s).
$$

Using (16) with $s$ replaced by $1 - s$, we get the required functional equation for $\zeta(s)$,

$$
\zeta(1 - s) = \frac{2\Gamma(s) \cos (\pi s/2)}{(2\pi)^s} \zeta(s).
$$

(17)

Since the right side of (17) is regular at $s = 1 - 2n\pi i/\log 2$, the only singularity of $\zeta(s)$ is the simple pole at $s = 1$. 

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