A COMBINATORIAL THEOREM WITH AN APPLICATION TO LATIN RECTANGLES

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1. Introduction. In the present paper a study is made of matrices of \( r \) rows and \( n \) columns, composed entirely of zeros and ones, with exactly \( k \) ones in each row. The problem considered is that of adjoining \( n-r \) rows of zeros and ones to obtain a square matrix with exactly \( k \) ones in each row and in each column. In \( \S 2 \) it is shown that the obvious necessary conditions for the adjunction of \( n-r \) rows are also sufficient. The theorem of \( \S 2 \) has an immediate application to the study of latin squares, and yields in \( \S 3 \) a generalization of the basic existence theorem of Marshall Hall \([2]\).\(^1\)

2. A combinatorial theorem.

**Theorem 1.** Let \( A \) be a matrix of \( r \) rows and \( n \) columns, composed entirely of zeros and ones, where \( 1 \leq r < n \). Let there be exactly \( k \) ones in each row, and let \( N(i) \) denote the number of ones in the \( i \)th column of \( A \). If, for each \( i = 1, 2, \ldots, n \),

\[
 k - (n - r) \leq N(i) \leq k,
\]

then \( n-r \) rows of zeros and ones may be adjoined to \( A \) to obtain a square matrix with exactly \( k \) ones in each row and in each column.

The proof is by mathematical induction. Let \( t \) denote the number of columns of \( A \) with \( N(i) < k \). Then \( n-t \) denotes the number of columns of \( A \) with \( N(i) = k \), and consequently \( kr = N(1) + \cdots + N(n) \geq (n-t)k + (k-(n-r))t \). Thus \( k(r-n) \geq t(r-n) \), whence \( t \geq k \).

Next let \( p \) denote the number of columns of \( A \) with \( N(i) = k - (n-r) \). Then \( n-p \) denotes the number of columns with \( N(i) > k - (n-r) \). Consequently \( kr = N(1) + \cdots + N(n) \leq p(k - (n-r)) + (n-p)k \), whence \( k(r-n) \leq p(r-n) \) and \( p \leq k \).

We now adjoin to \( A \) a row consisting of \( k \) ones and \( n-k \) zeros. Since \( t \geq k \), there are at least \( k \) positions where ones may be inserted so that the resulting \( (r+1) \)-rowed matrix will have at most \( k \) ones in each column. Moreover, since \( p \leq k \), the ones may be inserted in all of those columns with \( N(i) = k - (n-r) \). In the resulting \( (r+1) \)-rowed matrix, let \( M(i) \) denote the number of ones in the \( i \)th column.

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\(^1\) Numbers in brackets refer to the references at the end of the paper.
Because of the structure of the adjoined row, it is clear that

\[ k - (n - (r + 1)) \leq M(i) \leq k. \]

The process may be continued inductively, and the resulting square matrix possesses \( k \) ones in each row and column.

A rectangular matrix \( L \) composed of zeros and ones is called a permutation matrix provided that it satisfies the equation \( LL^T = I \), where \( L^T \) is the transpose of \( L \) and \( I \) is the identity matrix. Let \( A \) be a square matrix of zeros and ones, with exactly \( k \) ones in each row and in each column. A classical theorem of König asserts that

\[ A = L_1 + L_2 + \cdots + L_k, \]

where the \( L_i \) are permutation matrices [5]. Actually König’s theorem is a special case of P. Hall’s theorem on the distinct representatives of subsets [4]. The latter theorem has been the subject of the recent investigations of Everett and Whaples [1], and Marshall Hall [3].

**Corollary.** For the matrix \( A \) of Theorem 1, \( A = L_1 + L_2 + \cdots + L_k \), where the \( L_i \) are permutation matrices.

The corollary follows immediately upon and application of Theorem 1 and König’s theorem.

3. **The application to latin rectangles.** A latin rectangle of order \( r \) by \( s \) based upon the integers 1, 2, \cdots, \( n \) is defined as an array of \( r \) rows and \( s \) columns formed from the integers 1, 2, \cdots, \( n \) in such a way that the integers in each row and in each column are distinct. The latin rectangle is said to be extendible to an \( n \) by \( n \) latin square provided that it is possible to adjoin \( n - s \) columns and \( n - r \) rows in such a way that the resulting array is an \( n \) by \( n \) latin square. By utilizing the theory of distinct representatives of subsets, Marshall Hall has shown that every \( r \) by \( n \) latin rectangle may be extended to an \( n \) by \( n \) latin square [2].

**Theorem 2.** Let \( T \) be an \( r \) by \( s \) latin rectangle based upon the integers 1, 2, \cdots, \( n \). Let \( N(i) \) denote the number of times that the integer \( i \) occurs in \( T \). A necessary and sufficient condition in order that \( T \) may be extended to an \( n \) by \( n \) latin square is that for each \( i = 1, 2, \cdots, n \),

\[ N(i) \geq r + s - n. \]

Let \( T_i \) denote the set of \( s \) integers formed from the \( i \)th row of \( T \). Let \( S_i \) denote the set of the \( k = n - s \) integers among 1, 2, \cdots, \( n \) which are not in \( T_i \), and let \( M(i) \) denote the number of times that the integer \( i \) occurs among the sets \( S_1, S_2, \cdots, S_r \).
Now if $T$ is extendible to a latin square, then the integer $i$ cannot occur among the sets $S_1, S_2, \ldots, S_r$ more than $k = n - s$ times. Hence $M(i) \leq n - s$. But $N(i) + M(i) = r$, whence $N(i) \geq r + s - n$. Thus the condition of the theorem is a necessary one.

To prove the sufficiency we form from the sets $S_i$ a matrix $A$ of order $r$ by $n$, composed of zeros and ones. Let $S_i$ be composed of the integers $i_1, i_2, \ldots, i_k$. In the $i$th row of $A$ insert ones in columns $i_1, i_2, \ldots, i_k$, and zeros elsewhere in this row. The matrix $A$ has then exactly $k$ ones in each row, and $M(i)$ is now the sum of the $i$th column of $A$. By hypothesis $N(i) = r - M(i) \geq r + s - n$, so that for $i = 1, 2, \ldots, n$, $M(i) \leq k$. Since $T$ is an $r$ by $s$ latin rectangle, $N(i) \leq s$, whence $k - (n - r) \leq M(i)$. By the corollary of Theorem 1, it now follows that $A = L_1 + L_2 + \cdots + L_k$, where the $L_i$ are rectangular permutation matrices. Let the one in row $j$ of $L_i$ occur in column $t_j$. From the integers $t_j$ form the $k$ sets $(t_1, t_2, \ldots, t_r)$, each containing $r$ distinct integers. These sets may now be adjoined to $T$ to obtain a latin rectangle of order $r$ by $n$. The latter may then be extended to an $n$ by $n$ latin square as in [2]. This does not differ essentially from completing the transposed $n$ by $r$ latin rectangle to an $n$ by $n$ latin square by the method already indicated, the condition on $N(i)$ being then trivially satisfied.

**REFERENCES**


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