Let $S$ be a ring which satisfies a polynomial identity (in short: a PI-ring). The ring $S$ will be said to be a PI-ring of degree $d$ if $d$ is the minimal degree of the polynomial identities satisfied by $S$. We denote by $N = N(S)$ the radical of $S$, that is, the sum of all nilpotent ideals of $S$. Levitzki [1] has proved that a nil PI-ring of degree $d$ is an L-ring (that is, it coincides with its lower radical) and its length is bounded by $\log d/\log 2$. In the present note we show that the length of a nil PI-ring is not greater than 2 and that nil PI-rings of length 2 really exist. Even more, if $S$ is a nil PI-ring of degree $d$ then $S/N$ is a nilpotent ring whose index is bounded by $[d/2]$. This is a direct consequence of the following generalization of [1, Theorem 1]:

**Theorem 1.** If $S$ is a PI-ring$^2$ of degree $d$ and $T$ is a nil subring of $S$, then $T^n \subseteq N$ where $m = [d/2]$.

**Proof.** First we consider the case$^3$ where $T$ is a nilpotent subring of $S$. The proof of this case differs from the proof of [1, Theorem 1] only in that we consider a nilpotent subring instead of a single nilpotent element. That is, we consider the following subrings of $S$:

\[ A_{2i-1} = T^{n-i}ST^{i-1}, \quad A_{2i} = T^{n-i}ST^{i}, \quad i = 1, 2, \ldots, n, \]

where $n$ is an integer greater than $m = [d/2]$.

It is readily seen that $A_{\lambda}A_{\mu} \subseteq ST^nS$ if $\lambda > \mu$, hence

\[ A_{i_1}A_{i_2} \cdots A_{i_d} \subseteq ST^nS, \]

if $(i_1, \ldots, i_d)$ is a permutation of the $d$ letters $1, 2, \ldots, d$ which is not the identical permutation.

By (1) it follows also that

\[ A_1A_2 \cdots A_d = (T^{n-1}S)^d T^m. \]

We may assume by [1, Lemma 3] that $S$ satisfies the following identity$^4$:

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1 Numbers in brackets refer to the bibliography at the end of the paper.
2 It is not assumed that $S$ is a nil ring.
3 I am indebted to Levitzki for the present proof of this case.
4 Compare with (10) of [1]. For the conditions satisfied by the coefficients of the identity see [1, p. 335].
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(4) \[ x_1 x_2 \cdots x_d = \sum \beta^{-1} \beta(i)_x x_{i_1} \cdots x_{i_d}, \]
where the sum ranges over all permutations \(i\) of \(d\) letters, except the identical permutation.

From (2), (3), and from condition (II) of [1] satisfied by the coefficients of the identity (1), it follows by substituting \(x_i = a_i\) in (4) where \(a_i\) ranges over all elements of \(A_i\), \(i = 1, \cdots, d\), that

(5) \[ (T^mS)^d T^n \subseteq ST^n S. \]

Since \(T\) is nilpotent, there exists a smallest exponent \(n\) such that \(ST^n S\) is a nilpotent ideal. Suppose \(n > m\), then by (5) it follows easily that \((ST^{n-1} S)^{d+1} \subseteq ST^n S\), hence \(ST^{n-1} S\) is also nilpotent which is a contradiction to the minimality of \(n\). This completes the proof of the theorem in the case of nilpotent subrings.

We turn now to the general case. Let \(T\) be a nil subring of \(S\). By [1, Theorem 1] it follows that the quotient ring \((T, N)/N\) satisfies an identity of the form \(x^n = 0\), hence \((T, N)/N\) is semi-nilpotent (Kaplansky [3, Theorem 5] and Levitzki [1]). Since \(N\) is semi-nilpotent, the subring \((T, N)\) of \(S\) is also semi-nilpotent. Let \(t_1, \cdots, t_m\) be any \(m\) elements of \(T\), then the semi-nilpotency of \(T\) implies that the ring \(\{t_1, \cdots, t_m\}\) generated by these elements is nilpotent, hence by the preceding case \(\{t_1, \cdots, t_m\}^m \subseteq N\). Thus \(t_1 t_2 \cdots t_m \subseteq N\). Since this holds for any arbitrarily chosen elements of \(T\), we have \(T^m \subseteq N\). q.e.d.

**Remark.** By the preceding proof it follows that if \(T\) is a nilpotent subring of \(S\) of index \(p > m\), then \(ST^m S\) is a nilpotent ideal in \(S\). A more detailed application of (5) shows that \(1 + d + \cdots + d^{m-1}\) is an upper bound for the index of \(ST^m S\). Indeed by (5) we have \((ST^m S)^{d+1} \subseteq ST^{m+1} S\) and for the same reason \((ST^{m+1} S)^d ST^m \subseteq ST^{m+2} S\); hence \((ST^m S)^{d+1} \subseteq ST^{m+2} S\). By a successive application of (5), we obtain \((ST^m S)^{1+d+\cdots+d^{m-1}} \subseteq ST^n S = 0\),

which proves the remark.

By the preceding theorem, we have the following corollary.

**Corollary.** If \(S\) is a PI-ring of degree \(d\) such that its radical \(N\) is a nilpotent ideal of index \(p\), then the nil subrings of \(S\) are nilpotent rings of index not greater than \(p[d/2]\).

**Remark.** It has been shown recently [2] that the total matrix algebra of order \(n^2\) over a commutative field is a PI-ring of degree.

\* Apparently, [1, Theorem 1] is a special case of the preceding case of our theorem.
Hence by the preceding corollary it follows that the nil subrings of such algebras are nilpotent rings of index less than or equal to $n$. This is a special case of the well known result concerning nil subrings of rings which satisfy both chain conditions.

A simple consequence of Theorem 1 is:

**Theorem 2.** If $S$ is a nil PI-ring of degree $d$, then $S/N$ is a nilpotent ring whose index is bounded by $[d/2]$.

This implies that in this case $S$ is a nil PI-ring, $S = N_2(S)$, that is:

**Corollary.** A nil PI-ring is an L-ring of length less than or equal to $2$.

We conclude with an example of a nil PI-ring $S$ of degree $2n$ such that $S/N$ is a nilpotent ring whose index is $n$. This example shows that Theorem 2 provides a complete solution of the problem of the length of nil PI-rings and their structure modulo their radical.

We construct our example as follows: Let $R$ be a commutative ring with a unit such that its radical $N(R)$ is not nilpotent. Denote by $c_{ik}$, $i, k = 1, 2, \ldots, n$, an orthogonal base of a total matric algebra $R_n$ of order $n^2$ over $R$. One can easily generalise [2, Theorem 1] to total matric algebras over commutative rings and thus one obtains the result that $R_n$ satisfies a polynomial identity of degree $2n$ (that is, the standard identity $S_{2n}(x) = 0$). Our required ring $S$ is defined as the totality of the matrices $\sum \alpha_{ik} c_{ik}$ where $\alpha_{ik} \in N(R)$ for $i \geq k$. No restriction is imposed on the elements $\alpha_{ik}$, $i < k$, except that $\alpha_{ik} \in R$. Since $S \subseteq R_n$, it follows that $S$ is a PI-ring of degree less than or equal to $2n$. It is readily verified that $S^* \subseteq N(R)_n \subseteq N(S)$ where $N(R)_n$ is the totality of the matrices $\sum \beta_{ik} c_{ik}$, $\beta_{ik} \in N(R)$. Now consider the element $c = c_{11} + c_{22} + \cdots + c_{n-1,n}$. The ideals $c^i S$, $i = 1, 2, \ldots, n - 1$, contain a ring isomorphic to $N(R)$, that is, the ring of the matrices $c^i f = \rho c_{i+1,1}$ where $f = p c_{i+1,1}$, $\rho \in N(R)$. The latter ring is not nilpotent, hence $c^i N(S)$. This implies by [1, Theorem 1] that the degree of $S$ is greater than or equal to $2n$. This completes the proof that the ring $S$ has the required properties.

**Bibliography**