A GENERALIZATION OF THE RUTT-ROBERTS THEOREM

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One of the more useful theorems of plane topology was proved virtually simultaneously by Rutt [3] and Roberts [2]. As modified by Moore [1, p. 296], it is essentially this: Suppose that in the plane or the 2-sphere there exist two points, a and b; a collection, G, of continua whose union is a compact set, M, not containing a or b; and a continuum, C, such that the intersection of each two elements of G is precisely C. Then if no element of G separates a from b, neither does M. This is, of course, a form of addition theorem. Even in 3-space this result is not true as stated, for the collection of circles given in rectangular coordinates by $x^2+y^2+z^2-x=0$, $az=bx$, for all $a, b$, satisfies all the conditions on G with respect to the points $(1/2, 0, 0)$, $(2, 0, 0)$, whereas their union separates these points. There is, however, a theorem concerning linking, which is valid in quite general spaces, and which reduces to the above theorem in the plane.

**Theorem 1.** Let $S$ be a normal space acyclic in dimension $i+1$, and let $Z^i$ be a cycle on a compact subset $K$ of $S$. Let the compact set $M$ in $S-K$ be the union of a collection $G = \{C_a\}$ of closed sets satisfying the following: (1) for every $\alpha$, $Z^i \sim 0$ in $S-C_\alpha$; (2) there is a set $C$ which is the intersection of each two elements of $G$ and which links no $(i+1)$-cycle of $S$; (3) no closed set which is a union of elements of $G$ links any $(i+1)$-cycle; and (4) any closed set which is the union of more than one element of $G$ can be split into two closed proper subsets which are unions of elements of $G$. Then $Z^i \sim 0$ in $S-M$.

The relation of this to the original result is perhaps clear except for condition (4). It is not difficult to show (cf. Moore [1, p. 296]) that if in the original theorem $M$ separates $a$ from $b$, then (4) holds. The conditions (2) and (3) follow from the fact that no continuum links a 1-cycle in the 2-sphere.

**Proof.** Let $M'$ be a closed subset of $M$ which is the union of more than one element of $G$. Suppose that $Z^i$ does not bound in $S-M'$. I show that then $M'$ contains a proper closed subset $M''$, also the union of elements of $G$, such that $Z^i$ does not bound in $S-M''$. By

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1 The work on this paper was performed under the sponsorship of the ONR.
2 Numbers in brackets refer to the bibliography at the end of the paper.
3 Čech cycles and homologies on compact sets and with field coefficients are used throughout.
hypothesis (4), $M'$ is the union of two closed proper subsets, $M_1$ and $M_2$, both of which are unions of elements of $G$. The set $M_1 \cap M_2$ is either $C$ or a union of elements of $G$. Now suppose that neither $M_1$ nor $M_2$ links $Z^i$. Then for $k = 1, 2$, there is a chain $c_k^{i+1}$ in $S - M_k$, whose boundary is $Z^i$. Then $c_k^{i+1} - c_2^{i+1}$ is an $(i+1)$-cycle in $S - M_1 \cap M_2$, and by hypotheses (2) and (3) bounds there. But then from Wilder's generalization [5, p. 241] of the Alexander Addition Theorem, $Z^i$ must bound in $S - M'$. Hence at least one of $M_1$ or $M_2$ links $Z^i$; let that one be $M''$.

Now it is quite clear that if each of a monotone collection of compact sets links $Z^i$, then so does their intersection; and that the intersection of a monotone collection of closed unions of elements of $G$ is a union of elements of $G$, or is $C$. Hence if the theorem is false, by Zorn's lemma, there is a closed subset $M^*$ of $M$ which is irreducible with respect to the property of being a closed union of elements of $G$ that links $Z^i$. (The possibility that $M^* = C$ can be immediately discarded by hypothesis (1).) Since $M^*$ necessarily contains more than one element of $G$, it follows from the first paragraph of the proof that it is not a minimal closed union of elements of $G$ linking $Z^i$, thus yielding a contradiction.

The hypotheses of this theorem are disappointingly complex if one hopes for a theorem as useful in higher dimensions as the original has been in the plane. However, I have examples to show that none of the hypotheses can be removed, or indeed much relaxed, and still leave the theorem true. In particular, for each $n > 2$, I have an example in $S^n$ of a compact set carrying a nonbounding $(n-1)$-cycle, and which is the union of a collection $G$ of disjoint sets whose elements are points, arcs, and triods, with the property that for $0 < i < n-1$ no closed union of elements of $G$ carries a nonbounding $i$-cycle.

There is one case in which condition (4) can be replaced by rather natural conditions.

**Theorem 2.** Let the compact metric space $S$ be the union of a collection $G$ of closed sets such that there is a closed set $C$ which is the intersection of each two elements of $G$. Suppose that $G$ is upper semi-continuous in the sense that the union of all elements of $G$ intersecting a compact subset of $S - C$ is closed. Then $S$ is the union of two closed proper subsets, each a union of elements of $G$.

**Proof.** If $X$ is a compact set, not meeting $C$, and $H$ is the collection of all sets which are the intersection of $X$ with an element of $G$,
then $H$ is upper semi-continuous in the ordinary sense. Hence $H$ defines a continuous transformation $f: X \to Y$, where $Y$ is the decomposition space of $H$ (cf. Whyburn [4, pp. 125–127]). Given two proper closed subsets $A_1, A_2$ of $Y$, there are two proper closed subsets, $B_1, B_2$, of $Y$ such that $B_i$ contains $A_i$, and $B_1 \cup B_2 = Y$. The sets $f^{-1}(B_1), f^{-1}(B_2)$ are proper closed subsets of $X$ which are each unions of elements of $H$.

Now let $M_n$ denote the set of all points of $S$ at distance not less than $1/n$ from $C$, and let $H_n$ denote the collection of intersections of $M_n$ with elements of $G$. By the first paragraph, $M_1$ is the union of two closed proper subsets, $N_{11}$ and $N_{12}$, each a union of elements of $H_1$. The union of all elements of $H_2$ that contain points of $N_{1i}$, $i = 1, 2$, is closed, and is not all of $M_2$. Hence $M_2$ is the union of two closed proper subsets, $N_{21}$ and $N_{22}$, with $N_{21}$ containing $N_{1i}$, and each a union of elements of $H_2$. We similarly define $N_{21}, N_{22}, N_{31}, N_{32},$ and so on. Now $C \cup N_{21}$ and $C \cup N_{22}$ are both closed proper subsets of $S$, and each is a union of elements of $G$.

This last result perhaps has most interest when $C$ is empty and $G$ is the collection of point-inverses for some continuous transformation. By placing various conditions on such a transformation, Theorem 1 yields a number of theorems, none of which, however, seem to settle any of the major outstanding problems of topology.

Bibliography


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