TWO MAPPING PROPERTIES OF SCHLICHT FUNCTIONS

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The mapping properties we shall prove hold for the normalized exterior mapping function of a simple analytic curve. Let \( C \) be a simple analytic curve in the \( z \)-plane and designate its exterior by \( D \). The normalized exterior mapping function of \( C \) is the analytic function \( w = f(z) \) which is uniquely determined by the conditions that (i) it is regular in \( D \) except for a simple pole at \( z = \infty \), (ii) its power series expansion about \( z = \infty \) has the normalization

\[
(1) \quad w = z + a_0 + \frac{a_1}{z} + \cdots,
\]

and (iii) it maps \( D \) in a 1-1 manner onto the exterior of a circle \( \Sigma \), \( |w| = \rho \).

**Theorem I.** Let \( C \) be a simple analytic curve, and designate its exterior by \( D \). Let \( f(z) \) be the normalized exterior mapping function of \( C \). Let \( \sigma \) be a circle with center \( z_0 \), whose closed interior lies in \( D \). Then \( F(z) = f(z)/(z - z_0) \) maps \( \sigma \) onto a curve in the \( w \)-plane that is star-shaped from the point \( w = 0 \).

**Proof.** A curve \( \Gamma \) is star-shaped with respect to a point \( A \) in its interior if it is a simple curve, and if each point of \( \Gamma \) can be connected to \( A \) by a straight line lying in the interior of \( \Gamma \). Let \( \sigma \) have radius \( r \), and let \( Z \) be a point on \( \sigma \). Then \( Z - z_0 = re^{i\theta} \). Let \( F(Z) = Re^{i\phi} \). For the image of \( \sigma \) to be star-shaped, \( d\phi/d\theta \) must not vanish, and be of constant sign for \( 0 \leq \theta < 2\pi \). Since \( F(z) \) has a simple pole in \( \sigma \), and otherwise is regular and nonzero there, \( \phi \) decreases by \( 2\pi \) when \( \theta \) increases by \( 2\pi \), so \( d\phi/d\theta \) must be negative for some value \( \theta' \), \( 0 \leq \theta' < 2\pi \). We now show that it is negative for each value of \( \theta \) in the interval.

We first express \( d\phi/d\theta \) at a point \( Z \) on \( \sigma \) in terms of \( f(Z) \). Start with

\[
\frac{d\phi}{d\theta} = \frac{1}{d\theta} \text{Im} \log F(Z)
\]

(2)

\[
= \frac{d}{d\theta} \text{Im} ((\log f(Z) - \log (Z - z_0))).
\]

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Differentiate, to obtain

\[
\frac{d\phi}{d\theta} = \text{Im} \left( \frac{f'(Z)}{f(Z)} \frac{dZ}{d\theta} - \frac{1}{Z - z_0} \frac{dZ}{d\theta} \right).
\]

Substituting \(dZ/d\theta = i(Z - z_0)\), we obtain

\[
\frac{d\phi}{d\theta} = \text{Im} \left( i(Z - z_0) \frac{f'(Z)}{f(Z)} - i \right)
\]

\[
= \text{Re} \left( (Z - z_0) \frac{f'(Z)}{f(Z)} - 1 \right).
\]

We now use the Cauchy integral formula to obtain a representation for \(f'(Z)/f(Z)\). Since \(f'(z)/f(z)\) is regular in \(D\), and tends to zero as \(z \to \infty\), and since each point \(Z\) lies in \(D\), for a fixed \(Z\) we have

\[
\frac{f'(Z)}{f(Z)} = \frac{1}{2\pi i} \int_C \frac{f'(z)}{z - Z} \frac{1}{f(z)} \, dz.
\]

Let \(f(z) = \rho e^{ia}\) when \(z\) is on \(C\), and indicate the inverse of \(w = f(z)\) by \(z = z(w)\). Then \((f'(z)/if(z)) \, dz = d\alpha\) and from (5) we have

\[
\frac{f'(Z)}{f(Z)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{Z - z(\rho e^{ia})} \, d\alpha.
\]

Substituting (6) in (4), we obtain

\[
\frac{d\phi}{d\theta} = \text{Re} \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{Z - z_0}{Z - z(\rho e^{ia})} \, d\alpha - 1 \right).
\]

Since \((1/2\pi)\int_0^{2\pi} d\alpha = 1\), this can be written

\[
\frac{d\phi}{d\theta} = \text{Re} \left( \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{Z - z_0}{Z - z(\rho e^{ia})} - 1 \right) \, d\alpha \right)
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \text{Re} \left( \frac{z(\rho e^{ia}) - z_0}{Z - z(\rho e^{ia})} \right) \, d\alpha.
\]

The integrand in (8) is a continuous function of \(\alpha\) since the circumference of \(\sigma\) is bounded from \(C\). Hence, to prove \(d\phi/d\theta < 0\), it suffices to show that the integrand in (8) is negative for \(\alpha, 0 \leq \alpha < 2\pi\). Indeed, let \(\alpha_1\) be a value in this interval, and let \(z(\rho e^{i\alpha}) = z_1\). Then

\[
\text{Re} \frac{z_1 - z_0}{Z - z_1} < 0
\]

if
We write

\[
\Re \frac{Z - z_1}{z_1 - z_0} < 0.
\]

We write

\[
\Re \frac{Z - z_1}{z_1 - z_0} = \Re \frac{Z - z_0 + z_0 - z_1}{z_1 - z_0} = \Re \frac{Z - z_0}{z_1 - z_0} - 1.
\]

Since \(|Z - z_0| < |z_1 - z_0|\), we have

\[
\Re \frac{Z - z_0}{z_1 - z_0} - 1 \leq \left| \frac{Z - z_0}{z_1 - z_0} \right| - 1 < 0.
\]

From (12) it follows that the integrand in (8) is negative for each value of \(Z\) on \(\sigma\), and the theorem is proved.

**Theorem II.** Let \(C\) be a simple analytic curve, and designate its exterior by \(D\). Let \(f(z)\) be the normalized exterior mapping function of \(C\). Let \(Z_1\) and \(Z_2\) be two points in \(D\). If \(Z_1\) and each point of \(C\) lie on the same side of the perpendicular bisector, \(L\), of the line joining \(Z_1\) and \(Z_2\), then \(|f(Z_1)| < |f(Z_2)|\).

**Proof.** Representation (6) for \(f'(Z)/f(Z)\) is valid for \(Z\) in \(D\), hence by integration

\[
\log f(Z_i) = \frac{1}{2\pi} \int_{0}^{2\pi} \log (Z_i - z(\rho e^{i\alpha})) d\alpha \quad (i = 1, 2).
\]

It then follows that

\[
\log \left| \frac{f(Z_1)}{f(Z_2)} \right| = \frac{1}{2\pi} \int_{0}^{2\pi} \log \left| \frac{Z_1 - z(\rho e^{i\alpha})}{Z_2 - z(\rho e^{i\alpha})} \right| d\alpha.
\]

The integrand in (14) is a continuous function of \(\alpha\). From the hypothesis it follows that \(|(Z_1 - z(\rho e^{i\alpha}))/(Z_2 - z(\rho e^{i\alpha}))| \leq 1\). The inequality must hold for some \(\alpha\), for otherwise \(z(\rho e^{i\alpha})\) would be restricted to a line and \(C\) would not be simple. Hence the integral in (14) is negative and \(|f(Z_1)| < |f(Z_2)|\). This completes the proof.

Theorem I provides a complement to results stated by Pólya-Szegő \[1, pp. 104–105\]. It also gives part of the domain of schlichtness of the ratio of two schlicht functions.

At the suggestion of the referee we shall discuss Theorem II*, which is Theorem II under the more general hypothesis that \(D\) is an arbitrary, simply-connected domain containing \(z = \infty\), its bound-

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1 This refers to the bibliography at the end of the paper.
ary set $C$ is not an analytic curve (since this case is covered by Theorem II), and $f(z)$ is the analytic function with normalization (1) which maps $D$ in a 1-1 manner onto the exterior of a circle $\Sigma$, $|w| = \rho$. We lose no generality in taking $L$ to be the real axis, in which case $Z_1$ and $Z_2$ can be replaced by $Z$ and $Z^*$; where $\text{Im } Z > 0$.

If $C$ is in the half-plane $\text{Im } z > 0$, a level curve $C_1$ of $w = f(z)$ also lies in this half-plane with $Z$ in its exterior, and this curve can be used in place of $C$ in the proof used for Theorem II. This shows that $|f(Z)| < |f(Z^*)|$. If $C$ lies in the half-plane $\text{Im } z \geq 0$, touching $L$, the level curves of $f(z)$ will all be cut by $L$. Select a sequence, $\{C_k\}$, whose exteriors exhaust $D$. In the proof of Theorem II, replace $C$ by $C_n$ to obtain

$$\log \frac{|f(Z)|}{|f(Z^*)|} < \epsilon_n,$$

where $\epsilon_n > 0$ and $\epsilon_n$ tends to zero as $n$ tends to $\infty$. Hence

$$|f(Z)| \leq |f(Z^*)|.$$  

We now show that equality in (16) implies that $C$ coincides with $L$. Suppose $|f(Z_1)| = |f(Z_1^*)|$, where $Z_1$ is an interior point of $D$, satisfying $\text{Im } Z_1 > 0$. Since (16) holds in a neighborhood of $Z_1$, $f(z)$ must satisfy the functional relationship

$$f(z) = e^{i\alpha} (f(z^*))^*,$$

where $\alpha$ is a constant, $0 \leq \alpha < 2\pi$. Letting $x \to \infty$, $z = x + iy$, we see that, because of normalization (1), $\alpha$ must equal 0. But then, interpreted geometrically, (17), with $\alpha = 0$, implies that $C$ is symmetric about $L$. Since $C$ lies in the half-plane $\text{Im } z \geq 0$, it must coincide with $L$. Since $C$ is the boundary of a simply-connected domain, it is an interval of $L$. Hence

$$f(z) = \frac{(az + b) + ((az + b)^2 - 1)^{1/2}}{2a}$$

where $a$, $b$ are real, $a > 0$, and the branch is determined by choosing the positive sign of the radical for $z$ positive and sufficiently large.

Bibliography


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\* $Z^*$ denotes the complex conjugate of $Z$. 

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