A NECESSARY AND SUFFICIENT CONDITION THAT A CURVE LIE ON A HYPERQUADRIC

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Introduction. The purpose of this paper is to give a necessary and sufficient condition in terms of *the ordinary* curvatures of a curve for it to lie on *any* real central hyperquadric in a Euclidean space $\mathfrak{R}_n$. This problem was solved for the hypersphere in $\mathfrak{R}_n$ by Karel Havlíček in his paper *Contact des courbes et des hypersphères dans un espace euclidean à n dimensions.—Courbes sphériques,* which appears in Časopis pro Pěstování Matematiky a Fysiky vol. 72 (1947).

First we find a necessary and sufficient condition in terms of so-called hyperquadric curvatures for a curve to lie on a *given* real central hyperquadric. Then by means of invariant functions in §3 we find the solution of our problem.

1. **Fundamental concepts.** We shall begin by finding a necessary and sufficient condition that a curve lie on a hyperquadric homothetic to a given hyperquadric, where by a hyperquadric we always mean a real central hyperquadric whose equation may be written in the form

$$\sum_{1}^{n} a_{bc}(x^b - c^b)(x^c - c^c) - \gamma_0 d^2 = 0 \tag{1.1}$$

with given real coefficients $a_{bc} = a_{cb}$ and center $p(c^1, \cdots, c^n)$, where the determinant $[a_{bc}] = 1$, $\gamma_0 = \pm 1$, and d is a real number. If $d = 0$, then (1.1) is a hypercone with vertex p. If $d \neq 0$, then the number $|d|$ is called the radius of the hyperquadric (1.1). The coefficients a_{bc} are regarded here as the components of a metric tensor.

We shall use the notation $\bar{r}$ to denote the radius vector of a real curve $\bar{r} = \bar{r}(t)$, where by a curve we shall always mean a general curve defined in the following definition.

DEFINITION 1.1. A real curve $\mathfrak{C}$ of $\mathfrak{R}_n$ is called a general curve if it has the following properties:

(a) it is nonisotropic with respect to a_{bc};

(b) it has n different osculating spaces;

(c) its kth osculating space is not tangent to the isotropic hypercone of a_{bc} $(k = 1, \cdots, n-1)$.

By means of the metric a_{bc} we introduce all the usual notions of a tensor calculus. Hence for instance $\bar{u} \cdot \bar{v} \equiv a_{bc} u^b v^c$ is a scalar product of two vectors, and so on. In particular $\bar{u} \cdot \bar{v} = 0$ means that the vectors

Received by the editors August 14, 1950.

$\bar{u}$ and $\bar{v}$ are conjugate (with respect to our hyperquadric). The hyperquadric arc $\mathfrak{S}$ of $\mathfrak{C}$ is defined by $d\mathfrak{S}=(|d\bar{r}\cdot d\bar{r}|)^{1/2}$ where $\bar{r}=\bar{r}(t)$ are the equations of $\mathfrak{C}$. Henceforth we put $\bar{r}(t)=\overline{\mathfrak{R}}(\mathfrak{S})$ and denote derivatives with respect to $\mathfrak{S}$ by accents (that is, $\overline{\mathfrak{R}}'=d\overline{\mathfrak{R}}/d\mathfrak{S}$, and so on).

The Frenet formulas for $\mathfrak{C}$ (with respect to a_{bc}) are

$$\overline{\mathfrak{N}}_i' = -\gamma_i\gamma_{i+1}\mathfrak{K}_i\overline{\mathfrak{N}}_{i-1} + \mathfrak{K}_{i+1}\overline{\mathfrak{N}}_{i+1} \qquad (\mathfrak{K}_0 = \mathfrak{K}_n = 0) \tag{1.2}$$

where $\overline{\mathfrak{N}}_i$ $(i=0, 1, \cdots, n-1)$ are mutually hyperquadric conjugate unit vectors with $\overline{\mathfrak{N}}_i\cdot\overline{\mathfrak{N}}_i=\gamma_{i+1}$ $(\gamma_{i+1}=\pm 1)$ and $K_i>0$ $(i=1, \cdots, n-1)$ are the hyperquadric curvatures.

It may be shown that the K_i define a curve up to initial conditions. We shall designate by $\mathfrak{R}_i$ the radius of the hyperquadric curvature K_i (that is, $\mathfrak{R}_i=1/\mathfrak{K}_i$). The hyperplane spanned by the vectors $\overline{\mathfrak{N}}_i$ $(i=1, \cdots, n-1)$ at the point $\overline{\mathfrak{R}}(\mathfrak{S})$ of the curve $\overline{\mathfrak{R}}=\overline{\mathfrak{R}}(\mathfrak{S})$ will be termed the conjugate hyperplane of the curve.

2. **Contact of a hyperquadric and a curve.** Let $\bar{x}$ be the radius vector of a variable point in $\mathfrak{R}_n$. Let $\bar{p}$ be the radius vector of any fixed point and r any real numerical constant. Then any hyperquadric homothetic to (1.1) and having center $\bar{p}=(x_0^1, \cdots, x_0^n)$ and radius r has one of the equations

$$(\bar{x}-\bar{p})\cdot(\bar{x}-\bar{p}) - \gamma r^2 = 0, \quad \gamma = \pm 1. \tag{2.1}$$

If $r=0$ then this equation represents a hypercone. We shall never take the point $\overline{\mathfrak{R}}_0$ of the curve $\overline{\mathfrak{R}}=\overline{\mathfrak{R}}(\mathfrak{S})$ at the vertex of this hypercone.

DEFINITION 2.1. The power $\mathfrak{P}$ of an arbitrary point $\overline{\mathfrak{R}}$ in relation to the hyperquadric (2.1) is defined by the expression $\mathfrak{P}=(\overline{\mathfrak{R}}-\bar{p})\cdot(\overline{\mathfrak{R}}-\bar{p})-\gamma r^2$.

DEFINITION 2.2. Consider a hyperquadric (2.1) which goes through the point $\overline{\mathfrak{R}}_0$ of the curve $\overline{\mathfrak{R}}=\overline{\mathfrak{R}}(\mathfrak{S})$ and denote by $\mathfrak{P}(\mathfrak{S}, 0)$ the power of the point $\overline{\mathfrak{R}}(\mathfrak{S})$ of the curve with respect to the hyperquadric (2.1). Then if $\lim_{\mathfrak{S}=0}(\mathfrak{P}(\mathfrak{S}, 0)/\mathfrak{S}^p)=0$, $p=1, \cdots, q$, we say that the hyperquadric (2.1) has at the point $\overline{\mathfrak{R}}_0$ a contact with the curve $\overline{\mathfrak{R}}=\overline{\mathfrak{R}}(\mathfrak{S})$ of at least order q.

We may prove the following theorem.

THEOREM 2.1. *A necessary and sufficient condition that a hyperquadric* (2.1) *through the point* $\overline{\mathfrak{R}}_0$ *of a curve* $\overline{\mathfrak{R}}=\overline{\mathfrak{R}}(\mathfrak{S})$ *have a contact of at least order* $q=1$ *with the curve at this point is that its center lie in the conjugate hyperplane of the curve at the point* $\overline{\mathfrak{R}}_0$.

We shall discuss the significance of contact of order $q>1$ later in

Theorem 2.5. We may easily prove the next result.

THEOREM 2.2. *A necessary and sufficient condition that a curve lie on a hyperquadric* (2.1) *is that all its conjugate hyperplanes have at least one fixed common point and that this point is the center of the hyperquadric on which it lies.*

DEFINITION 2.3. Consider a curve $\overline{\mathfrak{R}}=\overline{\mathfrak{R}}(\mathfrak{S})$ and any fixed point $\mathfrak{p}$. Then we may write

$$\overline{\mathfrak{R}} - \mathfrak{p} = \sum_{i=0}^{n-1} \mathfrak{C}_i \overline{\mathfrak{R}}_i. \tag{2.2}$$

The functions $\mathfrak{C}_i$ $(i=0, 1, \cdots, n-1)$ will be called the relative curvatures of the curve $\overline{\mathfrak{R}}=\overline{\mathfrak{R}}(\mathfrak{S})$ with respect to the point $\mathfrak{p}$.

From (2.2) we have $\mathfrak{C}_i=\gamma_{i+1}(\overline{\mathfrak{R}}-\mathfrak{p})\cdot\overline{\mathfrak{R}}_i$ and by differentiation we obtain the equations:

$$\begin{aligned} \mathfrak{C}_0' &= \gamma_1\gamma_2\mathfrak{K}_1\mathfrak{C}_1 + 1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \\ \mathfrak{C}_i' &= -\mathfrak{K}_i\mathfrak{C}_{i-1} + \gamma_{i+1}\gamma_{i+2}\mathfrak{K}_{i+1}\mathfrak{C}_{i+1} \qquad (i = 1, \cdots, n-2), \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \\ \mathfrak{C}_{n-1}' &= -\mathfrak{K}_{n-1}\mathfrak{C}_{n-2}. \end{aligned} \tag{2.3}$$

From these we may show that the relative curvatures determine a curve up to initial conditions. From the equation $(\overline{\mathfrak{R}}-\mathfrak{p})\cdot(\overline{\mathfrak{R}}-\mathfrak{p}) = \sum_{i=0}^{n-1} \gamma_{i+1}\mathfrak{C}_i^2$ it is evident that the relative curvatures also determine the hyperquadric length of the vector $\overline{\mathfrak{R}}-\mathfrak{p}$.

THEOREM 2.3. *A necessary and sufficient condition that a curve* $\overline{\mathfrak{R}}=\overline{\mathfrak{R}}(\mathfrak{S})$ *lie on the hyperquadric* (2.1) *is that there exists a point* $\mathfrak{p}$ *such that*

$$\sum_{i=1}^{n-1} \gamma_{i+1}\mathfrak{C}_i\mathfrak{C}_i' = 0 \qquad (\mathfrak{C}_0 = 0) \tag{2.4}$$

at each of the points of the curve, where the $\mathfrak{C}$*'s are given by the recursion formulas*

$$\mathfrak{C}_i = \gamma_i\gamma_{i+1}\mathfrak{R}_i\mathfrak{C}_{i-1}' + \gamma_i\gamma_{i+1}\mathfrak{R}_i\mathfrak{K}_{i-1}\mathfrak{C}_{i-2} \qquad (i = 1, \cdots, n-1),\ \mathfrak{K}_0\mathfrak{C}_{-1} = -1. \tag{2.5}$$

The proof is based on the fact that $\sum_{i=0}^{n-1}\gamma_{i+1}\mathfrak{C}_i^2=\text{const.}$ is a necessary and sufficient condition for the curve to lie on (2.1) and on using equation(2.3).

For $n=3$ the condition (2.4) becomes $\gamma_2\mathfrak{R}_1+\gamma_3(\mathfrak{R}_1'\mathfrak{R}_2)'\mathfrak{R}_2=0$. If our given quadric is a sphere, then $\gamma_1=\gamma_2=\gamma_3=1$ and the $\mathfrak{R}$'s are the ordinary radii of curvatures. This condition then becomes the condition for a curve to be spherical.

THEOREM 2.4. *A necessary and sufficient condition that a curve* $\overline{\mathfrak{R}}=\overline{\mathfrak{R}}(\mathfrak{S})$ *lie on a hypercone* (*equation* (2.1) *with* $r=0$) *is that at least one of the equations*

$$(2.6)\qquad \begin{aligned} -\gamma_n\sum_{i=1}^{n-2}\gamma_{i+1}\mathfrak{C}_i(-\mathfrak{K}_i\mathfrak{C}_{i-1}+\gamma_{i+1}\gamma_{i+2}\mathfrak{K}_{i+1}\mathfrak{C}_{i+1}) \\ \pm\mathfrak{K}_{n-1}\mathfrak{C}_{n-2}\left(-\gamma_n\sum_{i=1}^{n-2}\gamma_{i+1}\mathfrak{C}_i^2\right)^{1/2}=0 \end{aligned}$$

is satisfied at each of the points of the curve.

This is a special case of Theorem 2.3 in which we make use of the equation $\sum_{i=1}^{n-1}\gamma_{i+1}\mathfrak{C}_i^2=0$.

THEOREM 2.5. *Let the hyperquadric* (2.1) *have a contact of at least order* q $(2\leqq q\leqq n)$ *with the curve* $\overline{\mathfrak{R}}=\overline{\mathfrak{R}}(\mathfrak{S})$ *at the point* $\overline{\mathfrak{R}}_0$. *Then the relative curvatures* $\mathfrak{C}_1,\cdots,\mathfrak{C}_{q-1}$ *are uniquely determined while the relative curvatures* $\mathfrak{C}_q,\cdots,\mathfrak{C}_{n-1}$ *are arbitrary.*

This theorem may be proved by induction on q and by using the formulas (1.2) in the equations for the derivatives of the power $\mathfrak{P}(\mathfrak{S})$.

3. The invariant parameter. We shall now derive a necessary and sufficient condition in terms of the ordinary curvatures of a curve that it lie on any real central hyperquadric.

We shall write $\bar{r}=\bar{r}(s)$ when the parameter used to describe the curve is the ordinary arc length s. All derivatives with respect to this parameter will be denoted by dots (that is, $d\bar{r}/ds=\bar{r}^{\cdot}$, and so forth). The ordinary tangent, normals, curvatures, radii of curvature, and so forth, will be denoted by small letters ($\bar{t}$, $\bar{n}_i$, k_i, ρ_i, and so forth).

When the parameter used to described the curve is the hyperquadric arc we shall write $\overline{\mathfrak{R}}=\overline{\mathfrak{R}}(\mathfrak{S})$ to denote the curve. The notation for the derivatives with respect to $\mathfrak{S}$ and for the hyperquadric tangent, normals, and so forth will remain the same as in the previous section of this paper.

Consider a function f of $\bar{r}(s)$, $\bar{t}(s)$, $\bar{n}_i(s)$, $k_i(s)$, ds, and their derivatives with respect to s. If we denote by $\mathfrak{F}$ the same function of $\overline{\mathfrak{R}}(\mathfrak{S})$, $\overline{\mathfrak{T}}(\mathfrak{S})$, $\overline{\mathfrak{N}}_i(\mathfrak{S})$, $\mathfrak{K}_i(\mathfrak{S})$, $d\mathfrak{S}$, and their derivatives with respect to $\mathfrak{S}$ and if $f=\mathfrak{F}$, then we say that f is an invariant.

THEOREM 3.1. *For the curve* $\bar{r}(s)=\overline{\mathfrak{R}}(\mathfrak{S})$ *we have the invariant* $\phi ds=\Phi d\mathfrak{S}$ *where* $\phi=(k_1^{n-1}k_2^{n-2}\cdots k_{n-2}^2k_{n-1})^{2/n(n+1)}$ *and* Φ *is the same function of the* $\mathfrak{K}$'s.

The proof follows immediately from transformation of parameter in the determinant $[\bar{r}^{\cdot}, \bar{r}^{\cdot\cdot}, \cdots, \bar{r}^{(n)}]$.

DEFINITION 3.1. The parameter σ defined by $d\sigma=\phi ds=\Phi d\mathfrak{S}$ will be referred to as the invariant parameter.

THEOREM 3.2. *The vectors* $\bar{a}_1, \cdots, \bar{a}_n$ *defined by* $\bar{a}_1=\bar{t}/\phi$, $\bar{a}_2=d\bar{a}_1/d\sigma, \cdots, \bar{a}_n=d\bar{a}_{n-1}/d\sigma$ *are invariant and linearly independent.*

They are seen to be invariant from the equation $\bar{a}_1=\bar{t}/\phi=\overline{\mathfrak{T}}/\Phi$. To show them linearly independent we first write

$$\bar{a}_i = \sum_{j=0}^{i-1} x_{ij}\bar{n}_j \qquad (\bar{n}_0 = \bar{t},\ i = 1, \cdots, n) \tag{3.1}$$

where the x_{ij} are obtained from the Frenet formulas as functions of k_i $(i=1, \cdots, n-1)$. Then if we form the determinant of their components we obtain

$$[\bar{a}_1, \cdots, \bar{a}_n] = \gamma \tag{3.2}$$

where from (3.1) $\gamma = \pm 1 = \operatorname{sgn} [\bar{t}, \bar{n}_1, \cdots, \bar{n}_{n-1}]$

which shows them to be linearly independent.

Now if we take the derivative of the determinant (3.2) we obtain $[\bar{a}_1, \bar{a}_2, \cdots, d\bar{a}_n/d\sigma]=0$ and therefore can write $d\bar{a}_n/d\sigma=\mu_1\bar{a}_1+\cdots+\mu_{n-1}\bar{a}_{n-1}$ where the μ_i are invariant. By substitution from (3.2) we obtain

$$\begin{aligned}
\mu_1 &= \gamma[d\bar{a}_n/d\sigma, \bar{a}_2, \cdots, \bar{a}_n],\\
\mu_2 &= \gamma[\bar{a}_1, d\bar{a}_n/d\sigma, \cdots, \bar{a}_n],\\
&\cdots\cdots\cdots\cdots,\\
\mu_{n-1} &= \gamma[\bar{a}_1, \bar{a}_2, \cdots, d\bar{a}_n/d\sigma, \bar{a}_n].
\end{aligned}$$

If we use the equations (3.1) we may compute the μ_i in terms of the ordinary curvatures of a curve and since the μ_i are invariant we may write

$$\mu_i(\mathfrak{R}_1, \cdots, \mathfrak{R}_{n-1}) = \mu_i(\rho_1, \cdots, \rho_{n-1}) \qquad (i = 1, \cdots, n-1). \tag{3.3}$$

The functions μ_i may also be shown to define a curve up to initial conditions and hence we may state the following theorem.

THEOREM 3.3. *The equations* (3.3) *may be looked upon as defining the* $\mathfrak{R}_i$ *in terms of the* ρ_i.

THEOREM 3.4. *A necessary and sufficient condition that a curve lie on a real central hyperquadric is that the equation* (2.4) *is satisfied at each of the points of the curve where the* $\mathfrak{C}_i$ *are given by equations* (2.5) *and* $\mathfrak{R}_1, \cdots, \mathfrak{R}_{n-1}$ *are to be computed by means of the equations* (3.3).

The proof is based on the result of Theorem 2.3 and using equations (3.3) as well as $d\mathfrak{S}=(\phi/\Phi)d\mathfrak{S}$ to eliminate the coefficients a_{bc} of the given hyperquadric from equation (2.4).

THEOREM 3.5. *A necessary and sufficient condition that a curve lie on any hypercone is that at least one of the equations* (2.6) *is satisfied at each of the points of the curve where the* $\mathfrak{C}_i$ *are given by equations* (2.5) *and* $\mathfrak{R}_1, \cdots, \mathfrak{R}_{n-1}$ *are to be computed by means of the equations* (3.3).

The proof is essentially the same as that of Theorem 3.4.

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