A BOOLEAN ALGEBRA WITHOUT PROPER AUTOMORPHISMS

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It is the purpose of this note to show that there exists an infinite Boolean algebra which has no proper automorphisms.\(^1\) We shall construct a simply ordered set \(S\), introduce a topology on this set in the usual manner, the so-called interval topology determined by the ordering relation,\(^2\) and prove that \(S\) is a compact zero-dimensional Hausdorff space and that the only homeomorphism on \(S\) onto \(S\) is the identity mapping. It is well known\(^3\) that the group of automorphisms of the set-field \(\mathcal{B}\) consisting of all open and closed subsets of \(S\) is isomorphic to the group of all homeomorphisms on \(S\) onto \(S\), whence it follows that \(\mathcal{B}\) has no proper automorphisms.

Consider a simply ordered set \(S\) with at least two elements. By the interval topology on \(S\) we mean the topology which has as a sub-basis for open sets, the family of all sets \(U \subseteq S\) such that either

\[
U = \{ x \mid x \in S \text{ and } x < a \}\quad \text{or}\quad U = \{ x \mid x \in S \text{ and } a < x \}
\]

for some \(a \in S\). These sets together with all sets of the form

\[
U = \{ x \mid x \in S \text{ and } a < x < b \}
\]

with \(a, b \in S\) constitute a basis for the interval topology on \(S\). We shall need the following theorem.

**Theorem 1.** If \(S\) is a simply ordered set, then the interval topology on \(S\) is a Hausdorff topology. In order for \(S\) to be compact and zero-dimensional, it is necessary and sufficient that the following conditions be satisfied:

(i) Every subset of \(S\) has a least upper bound and a greatest lower bound in \(S\).

(ii) Given any elements \(a, b \in S\) with \(a < b\), there exist elements \(x, y \in S\) such that \(a \leq x < y \leq b\), and such that \(x \leq u \leq y\) implies that \(u = x\) or \(u = y\).

The proof of this theorem offers no difficulty and will be omitted. We shall use certain familiar concepts and results pertaining to

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\(^2\) Cf. G. Birkhoff, loc. cit. p. 60.

simply ordered sets. Suppose $S$ is a simply ordered set, and let $\omega_\alpha$ and $\omega_\beta$ be regular initial ordinals (Anfangszahlen). An element $x \in S$ is said to have the character $(\omega_\alpha, \omega_\beta^*)$ if there exist a strictly increasing transfinite sequence $y = (y_0, y_1, \ldots, y_\xi, \ldots)$ $(\xi < \omega_\alpha)$ such that $x$ is the least upper bound of the elements $y_\xi$ and a strictly decreasing transfinite sequence $z = (z_0, z_1, \ldots, z_\eta, \ldots)$ $(\eta < \omega_\beta)$ such that $x$ is the greatest lower bound of the elements $z_\eta$. If $x$ is the least upper bound of a strictly increasing transfinite sequence $y = (y_0, y_1, \ldots, y_\xi, \ldots)$ and if the set of all elements $z \in S$ with $x < z$ is either empty or else has a smallest element, then $x$ is said to have the character $(\omega_\alpha, 0)$. The phrase “$x$ has the character $(0, \omega_\beta^*)$” is defined in a similar manner. Finally, $x$ is said to have the character $(0, 0)$ if the set of all elements $y \in S$ with $y < x$ is either empty or else has a largest element and if the set of all elements $z \in S$ with $x < z$ is either empty or else has a smallest element.

It is well known that each element $x \in S$ has one and only one character. We now prove the following theorem.

**Theorem 2.** Suppose $S$ is a simply ordered set, $f$ maps $S$ homeomorphically onto itself (in the interval topology on $S$), and $x \in S$. We then have:

(i) If $x$ has the character $(\omega_\alpha, \omega_\beta^*)$ where $\alpha \neq \beta$, then $f(x)$ has either the character $(\omega_\alpha, \omega_\beta^*)$ or $(\omega_\beta, \omega_\alpha^*)$.

(ii) If $x$ has one of the characters $(\omega_\alpha, \omega_\beta^*)$, $(\omega_\alpha, 0)$, $(0, \omega_\beta^*)$, then $f(x)$ also has one of these characters.

(iii) If $x$ has the character $(0, 0)$, then $f(x)$ also has the character $(0, 0)$.

**Proof.** Part (iii) is trivial since an element $x \in S$ has the character $(0, 0)$ if, and only if, $x$ is an isolated point in the interval topology of $S$.

Suppose $\omega_\alpha$ is a regular initial ordinal and assume that there exists a strictly increasing transfinite sequence $y = (y_0, y_1, \ldots, y_\xi, \ldots)$ $(\xi < \omega_\alpha)$ such that $x$ is the least upper bound of the elements $y_\xi$. Then $x$ belongs to the closure of the set $A = \{y_\xi \mid \xi < \omega_\alpha\}$.

Furthermore, if $x$ belongs to the closure of the set $A' \subseteq A$, then $A'$ must have $\omega_\alpha$ elements. Let

$$B = \{y \mid y \in A \text{ and } f(y) < f(x)\} \text{ and } C = \{z \mid z \in A \text{ and } f(x) < f(z)\}.$$
Then $A = B \cup C$, so that $x$ belongs to the closure of either $B$ or $C$. Suppose $x$ belongs to the closure of $B$, then $f(x)$ belongs to the closure of $f(B)$, whence it follows that $x$ is the least upper bound of $f(B)$. Hence there exists a strictly increasing transfinite sequence $y' = \langle y'_0, y'_1, \ldots, y'_\xi, \ldots \rangle \ (\xi < \omega_\alpha \leq \omega_a)$ with $y'_\xi \in f(B)$ such that $f(x)$ is the least upper bound of the elements $y'_\xi$. It follows that $\alpha' = \alpha$.

Similarly, if $x$ belongs to the closure of $C$, then there exists a strictly decreasing transfinite sequence $z = \langle z_0', z_1', \ldots, z_\eta', \ldots \rangle \ (\eta < \omega_a)$ with $z'_\eta \in f(C)$ such that $f(x)$ is the greatest lower bound of the elements $z'_\eta$.

From the above discussion we see that if $x$ has the character $(\omega_\alpha, \omega_\beta^*)$, then $f(x)$ has one of the characters $(\omega_\alpha, 0)$ or $(0, \omega_\beta^*)$ or else $f(x)$ has a character of the form $(\omega_\alpha, \omega^*_\alpha)$ or $(\omega_\beta, \omega^*_\beta)$. A similar argument shows that $f(x)$ must have one of the characters $(\omega_\beta, 0)$ or $(0, \omega_\alpha^*)$ or else $f(x)$ has a character of the form $(\omega_\beta, \omega^*_\beta)$ or $(\omega_\alpha, \omega^*_\alpha)$. Since $f(x)$ has only one character, we infer that if $\alpha \neq \beta$, then $f(x)$ must have either the character $(\omega, \omega^*_\beta)$ or $(\omega^*_\alpha, \omega_\alpha^*)$. Thus (ii) holds.

Now suppose $x$ has one of the characters $(\omega_\alpha, \omega^*_\beta)$, $(\omega_\alpha, 0)$, $(0, \omega^*_\alpha)$. Then $f(x)$ has either one of these three characters or else $f(x)$ has a character of the form $(\omega_\alpha, \omega^*_\beta)$ or $(\omega_\alpha, \omega^*_\alpha)$ with $\gamma \neq \alpha$. However, applying (i) with $f$ and $x$ replaced by $f^{-1}$ and $f(x)$, we see that in the latter case $x$ would have one of the characters $(\omega_\alpha, \omega^*_\beta)$, $(\omega_\beta, \omega_\alpha^*)$. Since this contradicts our assumption, the former case must apply, and (ii) holds. The proof is complete.

Theorem 3. There exists an infinite compact zero-dimensional topological space $S$ such that the only homeomorphism on $S$ onto $S$ is the identity mapping.

Proof. Let $\kappa_0 = 0$ and $\kappa_{n+1} = \omega_{\kappa_n}$ for $n = 0, 1, \ldots$, and for each $k < \omega_0$ let $A_k$ be the family of all sequences $x = \langle x_0, x_1, \ldots, x_n, \ldots \rangle$ of ordinals $x_n$ such that $x_n < \kappa_{n+1}$ for $n = 0, 1, \ldots, k$ and $x_n = 0$ for $n = k+1, k+2, \ldots$. Then $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots$, and the cardinal number of $A_k$ is $\aleph_{\kappa_k}$. The set $B_k$ of all ordinals $\alpha$ with $\kappa_k \leq \alpha < \kappa_{k+1}$ also has $\aleph_{\kappa_k}$ elements, hence there exists a univalent function $f_k$ on $A_k$ to $B_k$.

We next construct for each $k < \omega_0$ a subset $C_k$ of $A_k$. Let $C_0 = A_0$, and, assuming that $C_k$ has already been defined, consider a sequence $x \in A_{k+1}$. Then the sequence $x' = \langle x_0, x_1, \ldots, x_k, 0, 0, \ldots \rangle$ is a member of $A_k$. We let $x \in C_{k+1}$ if and only if $x' \in C_k$ and $x_{k+1} < \omega_a$ where $\alpha = f_k(x') + 1$. Thus $C_k$ is defined for each $k < \omega_0$; we denote the set-theoretical union of all the sets $C_k$ by $C$.

Let $\leq$ be the lexicographic ordering of $C$; that is, given two se-
sequences \( x, y \in C \), we let \( x \leq y \) if, and only if, \( x = y \) or else \( x_k < y_k \) where \( k \) is the smallest natural number such that \( x_k \neq y_k \). This relation simply orders \( C \) and well orders each of the sets \( C_k \).

Let \( D_0 \) be the set of all ordered pairs \( \langle x, 0 \rangle \) with \( x \in C \) such that \( x \neq \langle 0, 0, 0, \cdots \rangle \) and the last nonzero term of \( x \) is not a limiting ordinal, let \( D_1 \) be the set of all ordered pairs \( \langle x, 1 \rangle \) with \( x \in C \), and let \( D = D_0 \cup D_1 \). For two ordered pairs \( \langle x, i \rangle, \langle y, j \rangle \) in \( D \) we write \( \langle x, i \rangle \leq \langle y, j \rangle \) if, and only if, either \( x < y \) or else \( x = y \) and \( i \leq j \).

The simply ordered set \( D \) can be imbedded in a simply ordered set \( S \) such that every subset of \( S \) has a least upper bound and a greatest lower bound in \( S \), and such that each element \( p \in S \) is the least upper bound of all elements \( q \in D \) with \( q \leq p \) and the greatest lower bound of all elements \( r \in D \) with \( r \leq p \). We shall prove that the interval topology on \( S \) satisfies the conditions of the theorem.

The set \( C \) is dense in itself. In fact, suppose \( x, y \in C \) and \( x < y \). If \( n \) is the smallest natural number such that \( x^n \neq y^n \), then \( x_n < y_n \), the sequence \( z = \langle x_0, x_1, \cdots, x_n, x_{n+1}+1, 0, 0, \cdots \rangle \) is a member of \( C \), and \( x < z < y \).

Suppose \( p, q \in S \) and \( p < q \). Then \( p \leq r < \delta \leq q \) for some \( r, \delta \in D \), and we have \( r = \langle x, i \rangle \) and \( \delta = \langle y, j \rangle \) with \( x, y \in C \) and \( i, j = 0, 1 \). If \( x = y \), then \( i = 0 \) and \( j = 1 \), and there exists no element \( t \in S \) with \( r < t < \delta \). If \( x < y \), then \( x < z < y \) for some \( z \in C \). First suppose the last nonzero term \( z_n \) of \( z \) is a limiting ordinal. Then \( z \) is the least upper bound of the sequences \( \langle z_0, z_1, \cdots, z_{n-1}, \alpha+1, 0, 0, \cdots \rangle \) with \( \alpha < z_n \). Hence one of these sequences must be larger than \( x \). We therefore see that \( z \) can be so chosen that the last nonzero term is not a limiting ordinal. Letting \( r' = \langle z, 0 \rangle \) and \( \delta' = \langle z, 1 \rangle \), we thus have \( r', \delta' \in D \subseteq S \) and \( p \leq r' < \delta' \leq q \), and there exists no element \( t \in S \) with \( r' < t < \delta' \). We conclude by Theorem 1 that \( S \) is compact and zero-dimensional.

From the discussion in the preceding paragraph it is clear that the set \( D_0 \) is everywhere dense in \( S \) in the interval topology. In order to prove that the only homeomorphism on \( S \) onto \( S \) is the identity mapping, it is therefore sufficient to show that every homeomorphism on \( S \) onto \( S \) maps each member of \( D_0 \) onto itself. For this purpose we study the characters of the elements of \( S \).

Suppose \( p \in D_0 \). Then \( p = \langle x, 0 \rangle \) where \( x \in C \) and the last nonzero term \( x_n \) of \( x \) is not a limiting ordinal. Hence \( x_n = u+1 \) for some ordinal \( u \). The sequence \( x' = \langle x_0, x_1, \cdots, x_{n-1}, u, 0, 0, \cdots \rangle \) is then a member of \( C \). Letting \( \alpha = f_n(x') + 1 \) and \( y^\alpha = \langle x_0, x_1, \cdots, x_{n-1}, u, \beta, 0, 0, 0, \cdots \rangle \) for every \( \beta < \omega_n \), we see that all the sequences \( y^\beta \) are members of \( C \), and that their least upper bound is \( x \). Hence \( p \) is the least upper bound of all the ordered pairs \( \langle y^\beta, 1 \rangle \) with \( \beta < \omega_n \), and
we conclude that $p$ has the character $(\omega_a, 0)$. Recalling how the functions $f_n$ were chosen, we see that distinct members of $D_0$ have different characters.

Suppose $p \in D_1$. Then $p = (x, 1)$ with $x \in C$. Choosing $n$ sufficiently large we have $x_n = x_{n+1} = \cdots = 0$. For each $k < \omega_0$ we let $\xi^k = (\xi_0^k, \xi_1^k, \cdots, \xi_m^k, \cdots)$ where $\xi_m^k = x_m$ for $m < n$, $\xi_n^k = 1$, and $\xi_m^k = 0$ for $n \leq m \neq n + k$. Then $\xi^0 > \xi^1 > \cdots > \xi^k \cdots$, and the greatest lower bound of the sequences $\xi^k$ is $x$. Hence the greatest lower bound of the ordered pairs $(\xi^k, 1)$ is $p$. We thus see that $p$ has either the character $(0, \omega_0^*)$ or else a character of the form $(\omega_a, \omega_0^*)$.

Suppose $p \in S - D$. First assume that $p$ is the largest element of $S$. Observe that the sequences $y^k = (k, 0, 0, \cdots)$ with $k < \omega_0$ have no upper bound in $C$. Therefore the least upper bound of the ordered pairs $(y^k, 1)$ is $p$, whence it follows that the character of $p$ is $(\omega_0, 0)$. Next suppose $p$ is not the largest element of $S$. Then the set $E$ of all elements $q \in D_1$ with $p < q$ is nonempty. In fact, if we let $E_k$ be the set of all ordered pairs $q = (x, 1)$ with $x \in C_k$, then $E \cap E_0$ is nonempty and hence each of the sets $E \cap E_k$ is nonempty. Since $E_k$ is well ordered, each of the sets $E \cap E_k$ has a smallest element $q_k$. Hence $q_0 \geq q_1 \geq \cdots \geq q_k \geq \cdots$, and the greatest lower bound of the elements $q_k$ is $p$. Hence $p$ has a character of the form $(\omega_a, \omega_0^*)$.

Suppose $\phi$ is a homeomorphism on $S$ onto $S$, and consider an element $p \in D_0$. Then $p$ has a character of the form $(\omega_a, 0)$ with $\alpha > 0$, whence it follows by Theorem 2 that $\phi(p)$ has one of the characters $(\omega_a, 0)$, $(0, \omega_0^*)$, $(\omega_a, \omega_0^*)$. But no other member of $S$ has one of these characters, so that we must have $\phi(p) = p$. Since this holds for every $p \in D_0$, we conclude that $\phi$ is the identity mapping. The proof is complete.

We now obtain the following theorem.

**Theorem 4.** There exists an infinite Boolean algebra which has no proper automorphisms.

**Proof.** This is proved by Theorem 3 and the introductory remarks.

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