A GENERAL DEPENDENCE RELATION FOR LATTICES

DANIEL T. FINKBEINER

1. Introduction. An arithmetic theory of lattices, which concerns the representation of lattice elements in terms of irreducible elements, has been developed only for lattices which satisfy the semi-modular law and a chain condition. For such a lattice \( \mathfrak{L} \) the principal results are:

1) \( \mathfrak{L} \) has unique irredundant decompositions if and only if \( \mathfrak{L} \) is locally distributive \([1]\).
2) The number of components in an irredundant decomposition is unique if and only if \( \mathfrak{L} \) is locally modular \([2]\).

However, the uniqueness of the number of components does not imply semi-modularity, and a characterization of lattices which have this uniqueness property will require an arithmetic theory for a more extensive class of lattices.

A natural method of approach to such a theory is to imbed an arbitrary lattice in a semi-modular lattice. However, the usual method of imbedding by means of dependence relations produces lattices in which the join irreducibles are points and the meet irreducibles are maximal elements. Clearly this is not suitable for arithmetic considerations. The purpose of this note is to formulate the properties a dependence relation must have if the imbedding is to preserve join irreducibility and if the imbedding lattice is to be semi-modular. Since the join irreducibles form a partially ordered set \( Q \), the dependence relation will be defined on subsets of \( Q \).

2. Terminology and notation. The lattice theoretic terms used here are standard \([3]\). Unless otherwise noted, irreducible means join-irreducible. The null element is trivially irreducible and is excluded from sets of irreducibles. Lattice operations are denoted \( \lor \) and \( \land \), and the corresponding set operations by \( \lor \) and \( \land \). If \( q \) is an element of a partially ordered set \( Q \), let \( S_q = \{ q' \in Q \mid q' \subseteq q \} \).

3. Properties of the dependence relations. A relation \( D \) between the elements and subsets of a set \( Q \) is called a dependence relation
provided $D$ satisfies

(D1) $q \in D S \cap q$ for arbitrary $S \subseteq Q$,

(D2) $q \in D S$ and $S \subseteq T$ implies $q \in D T$.

The closure of $S$ is defined by $C(S) = \{ q \in Q \mid q \in D S \}$. The closed subsets of $Q$ form a complete lattice $\mathcal{L}'$, and $\mathcal{L}'$ is a semi-modular point lattice provided $D$ also satisfies the exchange axiom

(D3) $q \in D S \cap q'$ implies either $q \in D S$ or $q' \in D S \cap q$.

Now let $\mathfrak{P}$ be a partially ordered set, and let $\Delta$ be a relation between the elements and subsets of $Q$. The following properties of $\Delta$ are considered.

(A1) $q \in \mathfrak{P} q$ implies $q' \in D S \cap q$ for arbitrary $S \subseteq Q$.

(A2) $q \in A S$ and $S \subseteq T$ implies $q \in A T$.

(A3) $q \in A S$ implies $q' \subseteq q$.

(A4) $q \in A S$ and $S \subseteq T$ implies $q \subseteq S$.

(A5) If $q' \in q \subseteq q'$ implies $q' \subseteq S$, then $q \subseteq S$ or $q' \subseteq S \cap q$.

Any relation which satisfies $A1$ and $A2$ is a dependence relation, and the closed sets form a complete lattice $\mathcal{L}'$. The following lemmas are proved readily.

**Lemma 1.** If $\Delta$ satisfies $A1-A3$, then $S_q$ is closed for all $q \in Q$.

**Lemma 2.** If $\Delta$ satisfies $A1-A3$ and if $S$ is completely irreducible in $\mathcal{L}'$, then $S = S_q$ for some $q \in Q$.

**Lemma 3.** If $\Delta$ satisfies $A1-A3$, $S_q$ is completely irreducible in $\mathcal{L}'$ for every $q \in Q$ if and only if $A4$ is also satisfied.

The first lemma implies that the natural mapping $q \to S_q$ is an imbedding of $Q$ into $\mathcal{L}'$. Clearly $S_q^r \subseteq S_q$ if and only if $q' \subseteq q$, and if equality holds in either relation, it holds in both. This gives the following result.

**Theorem 1.** Let $\Delta$ satisfy $A1-A3$. The set $Q'$ of completely irreducible elements of $\mathcal{L}'$ is isomorphic to $Q$ under the natural mapping if and only if $A4$ is also satisfied.

Now consider the question of semi-modularity. MacLane [4] has formulated an exchange axiom (E6) which is free from any covering assumptions and which implies semi-modularity.

(E6) If $S \cap T \subseteq R \subseteq S \cup T$, then $S_1$ exists such that $S \cap T \subseteq S_1 \subseteq S$, and $(R \cup S_1) \cap T \subseteq T$.

**Theorem 2.** If $\Delta$ satisfies $A1$, $A2$, and $A5$ and if $Q$ satisfies the de-

*By $S \cap T$ we mean $q \in D T$ for all $q \in S$. 

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descending chain condition, then \( \mathcal{Q}' \) satisfies E6 and therefore is semi-modular.

**Proof.** Let \( S, T, \) and \( R \) be closed sets such that \( S \cap T \subseteq R \subseteq S \cup T \). The descending chain condition implies the existence of \( q'_i \in S - S \cap T \) such that \( q'' \subseteq q'_i \) implies \( q'' \Delta S \cap T \). Clearly \( q'' \Delta R \) for all \( q'' \subseteq q'_i \). Let \( q_i \in T - R \); then \( q_i \Delta R \). If \( q_i \Delta R \cap q'_i \), then by A5 \( q'_i \Delta R \cap q_i \), and therefore \( (S \cap T) \cap q'_i \Delta R \cap q_i \). Let \( S_1 = C[(S \cap T) \cap q'_i] \subseteq S \). Then \( S_1 \subseteq C(R \cap q_i) \subseteq T \), and \( S_1 \subseteq S \cap T \) which contradicts the choice of \( q'_i \). Hence for all \( q_i \in T - R, q_i \Delta R \cap q'_i \) and \( C(R \cap q'_i) \cap T \subseteq T \). But also \( C(R \cup S_1) = C(R \cup q'_i) \), and finally \( (R \cup S_1) \cap T \subseteq T \).

It should be observed that Theorem 2 implies the classical theorem concerning dependence relations and semi-modular point lattices [4] when \( Q \) is an unordered set. Theorems 1 and 2 combine to give the main result.

**Theorem 3.** Let \( Q \) be a partially ordered set in which the descending chain condition holds. If \( \Delta \) satisfies \( \Delta_1 - \Delta_5 \), the lattice \( \mathcal{Q}' \) of closed subsets of \( Q \) is a complete semi-modular lattice. The set of completely irreducible elements of \( \mathcal{Q}' \) is isomorphic to \( Q \), and every element of \( \mathcal{Q}' \) is the union of such elements.

As an example, let \( Q \) be the set of irreducible elements of a lattice which satisfies the descending chain condition. Define the relation

(A) \( q \Delta S \) if and only if \( q \subseteq S \).

It is easy to verify that (A) satisfies \( \Delta_1 - \Delta_4 \), and that the mapping \( b \rightarrow S_b \) imbeds \( \mathcal{Q} \) isomorphically onto \( \mathcal{Q}' \). If (A) satisfies \( \Delta_5 \), \( \mathcal{Q}' \) is semi-modular; hence so is \( \mathcal{Q} \). The converse is also true; if \( \mathcal{Q} \) is semi-modular, (A) satisfies \( \Delta_5 \).

It is interesting to note that some restriction of \( Q \), such as the chain condition, is necessary in order that \( \Delta_5 \) imply the semi-modularity of \( \mathcal{Q}' \). Consider the lattice defined by

\[
\begin{align*}
&u > t > s > q_1 > q_2 > \cdots > z, \\
&u > r > p_1 > p_2 > p_3 > \cdots > z,
\end{align*}
\]

where every element except \( u \) and \( r \) is completely irreducible. Relation (A) satisfies \( \Delta_1 - \Delta_5 \) when defined on \( \mathcal{Q} \), but \( \mathcal{Q}' \) is isomorphic to \( \mathcal{Q} \) and is not semi-modular.

Other specific examples of irreducibility-preserving dependence re-

\footnote{\( \Delta \) denotes the denial of \( \Delta \).}
lations which have been applied to lattice imbedding problems will be given elsewhere.

References


California Institute of Technology and
Yale University