ON THE INTEGRATION SCHEME OF MARÉCHAL

EMIL GROSSWALD

J. E. Wilkins, Jr.\(^1\) proves the following assumption of A. Maréchal:\(^2\)

Let \(f(x, y)\) be a function of 2 variables, continuous in the interior of the circle \(C\), of radius \(R\); then

\[
\int \int \limits_C f(x, y) \, d\sigma = \lim_{\epsilon \to 0} \left\{ 2\pi a \int \limits_{S_a} f(x, y) \cdot ds \right\},
\]

where the double integral extends over the area of \(C\) and the line integral is taken along the arc of the archimedean spiral \((S_a)\) interior to \(C\).

In what follows, we give a short, elementary proof of (1), and two extensions.

I. Proof of (1). Let \(x = r \cos \phi\), \(y = r \sin \phi\) and use the notations:

\[
A(m, n) = \frac{1}{2\pi} \int_0^{2\pi} r^m \cos^m \phi \sin^n \phi \, d\phi,
\]

\[
B(m, n) = \frac{1}{\pi} \int_0^{2\pi} \cos^m \phi \sin^n \phi \, d\phi,
\]

\[
C(m) = \int_0^R r^m \, dr.
\]

Then, in (1), \(d\sigma = rd\sigma r\, d\phi\) and \(ds = (r^2 + a^2)^{1/2} \cdot d\phi\).

As any continuous function can be approximated by a uniformly convergent sequence of polynomials, it is sufficient to prove (1) for \(f(x, y) = x^m y^n = r^{m+n} \cos^m \phi \sin^n \phi\). If \(m = n = 0\), (1) is verified by direct integration. If \(m + n > 0\), the first member of (1) becomes

\[
\int_0^R \int_0^{2\pi} r^{m+n+1} \cos^m \phi \sin^n \phi \, r \, dr \, d\phi = 2\pi A(m, n) \cdot C(m + n + 1);
\]

and the second member may be written as

\[
\text{Received by the editors September 27, 1950.}
\]


706
ON THE INTEGRATION SCHEME OF MARÉCHAL 707

\[ \lim_{a \to 0} \left\{ 2\pi a \int_{S_a} r^{m+n}(r^2 + a^2)^{1/2} \cos^m \phi \sin^n \phi d\phi \right\} \]

\[ = \lim_{a \to 0} 2\pi \int_0^R \left\{ r^{m+n+1} \cos^m (r/a) \cdot \sin^n (r/a) + O(a) \right\} dr. \]

In the (finite) Fourier expansion \( \cos^m (r/a) \cdot \sin^n (r/a) = a_0/2 + \sum_{r=1}^{m+n} (a_r \cos (r/a) + b_r \sin (r/a)) \) the first term is \( a_0/2 = (1/2\pi) \int_0^\infty \cos^m \phi \sin^n \phi d\phi = A(m, n) \). When \( a \to 0 \), then \( v/a \to \infty \), so that, by the Riemann-Lebesgue lemma on Fourier series,

\[ \lim_{a \to 0} \int_0^R r^{m+n+1} \cos (v/a) dr = 0, \lim_{a \to 0} \int_0^R r^{m+n+1} \sin (v/a) dr = 0, \]

and (5) reduces to

\[ \lim_{a \to 0} \left\{ 2\pi \int_0^R r^{m+n+1} A(m, n) dr + 2\pi R \cdot O(a) \right\} = 2\pi A(m, n) \cdot C(m+n+1), \]

the same as (4), proving (1).

II. Consider the integral of the continuous function \( f(x, y, z) \) taken on the surface of the sphere \( S \) of radius \( R \). We may approximate it by the integral taken on a narrow strip, winding around the sphere, along the path

\[ R\phi = a\theta, \]

from one pole \( (\phi=\theta=0) \) to the other \( (\phi=\pi, \theta=R\pi/a) \). We take the width of the strip to be \( 2\pi a \) and then make \( a \) tend to zero. The relation similar to (1) which we want to prove is, therefore,

\[ \int \int_{S'} f(x, y, z) d\sigma = \lim_{a \to 0} \left\{ 2\pi a \int_{S_a'} f(x, y, z) ds \right\}. \]

As before, it is sufficient to prove (6) for \( f(x, y, z) = x^m y^n z^k, m+n+k > 0 \), because, for \( m=n=k=0 \), \( f(x, y, z) = 1 \) and (6) is verified by direct integration. The first member of (6) becomes successively, using (3),

\[ \int \int_{S} R^{m+n+k} \sin^{m+n} \phi \cos^k \phi \cos^m \theta \sin^n \theta \cdot R^2 \sin \phi d\phi d\theta = R^{m+n+k+2} \int_0^{2\pi} \cos^m \theta \cdot \sin^n \theta \cdot \sin \phi d\phi d\theta \]

\[ = 2\pi A(m, n) \cdot \pi B(k, m+n+1) = 2\pi^2 R^{m+n+k+2} A(m, n) \cdot B(k, m+n+1). \]

The second member of (6) becomes, as under I,

\[ \lim_{a \to 0} \left\{ 2\pi a \int_{S_a'} R^{m+n+k} \sin^{m+n} \phi \cos^k \phi \cos^m \theta \sin^n \theta \cdot (R \sin \phi + O(a)) d\theta \right\} \]

\[ = \lim_{a \to 0} \left\{ 2\pi a \cdot R^{m+n+k+1} \int_{S_a'} \sin^{m+n+1} \phi \cos^k \phi \cos^m \theta \sin^n \theta d\theta + O(a) \right\}. \]
By (2'), \( \theta = \phi R/a \), so that the last expression becomes

\[
(8) \quad \lim_{a \to 0} \left\{ 2\pi R^{m+n+k+2} \int_0^\pi \sin^{m+n+1} \phi \cos^k \phi \cos^m (R\phi/a) \cdot \sin^n (R\phi/a) d\phi \right\}.
\]

Here \( g(\phi) = \sin^{m+n+1} \phi \cos^k \phi \) is a continuous, bounded function and we use, as under I, the relation

\[
\cos^m (R\phi/a) \cdot \sin^n (R\phi/a) = a_0/2 + \sum_{r=1}^{m+n} (a_r \cos (R\phi/a) + b_r \sin (R\phi/a))
\]

with \( a_0/2 = A(m, n) \). When \( a \to 0 \), \( R\phi/a \to \infty \) and it follows from the Riemann-Lebesgue lemma that all the expressions of the form

\[
\lim_{a \to 0} \int_0^\pi g(\phi) \cos (R\phi/a) d\phi \quad \text{and} \quad \lim_{a \to 0} \int_0^\pi g(\phi) \sin (R\phi/a) d\phi,
\]

\( \nu = 1, 2, \ldots, m + n \),

vanish and (8) reduces to

\[
2\pi R^{m+n+k+2} A(m, n) \int_0^\pi \sin^{m+n+1} \phi \cos^k \phi d\phi
\]

\[
= 2\pi^2 R^{m+n+k+2} A(m, n) \cdot B(k, m + n + 1),
\]

same as (7), proving (6).

III. Let the sphere \( S^r \), of radius \( r \), be covered by a wire of square section \( 2\times a \times 2\times a \), winding on the sphere along a spiral like \((S^r)\). The outer surface of the wire is a new sphere of radius \( r_{v+1} = r_v + 2\pi a \) and let this be covered in the same way, by the same wire, and so forth. In particular, making \( a \to 0 \), we can fill the interior of the sphere \( S \), of radius \( R \), with such successive layers of wire, winding along spirals of equations

\[
(2'') \quad (S^r) \quad r_v \phi = a\theta, \quad \nu = 1, 2, \ldots, \lfloor R/2\pi a \rfloor,
\]

where, in the \( \nu \)th layer from the center, \( r_v = \nu(2\pi a) \). We may attempt to approximate an integral, extended over the volume of the sphere, by the sum of integrals taken along the \((S^r)\), which wind around the successive spherical shells, and are led to consider the equality

\[
\int \int \int_{S^r} f(x, y, z) dr = \lim_{a \to 0} \left\{ 4\pi^2 a^2 \sum_{r=1}^{\lfloor R/2\pi a \rfloor} \int_{S^r} f(x, y, z) \cdot ds \right\},
\]

where the integral in the first member is extended over the volume of

\[ \lfloor R/2\pi a \rfloor \] stands for the largest integer not exceeding \( R/2\pi a \).
S and the integrals of the second member are taken along the arcs $s^{(b)}$ of the spirals $(S_\psi)$ from $(2^n)$. The proof, proceeding along the same lines as that of (6), is suppressed here.

**Remark.** The explicit values of the elementary integrals (3) are, of course, well known; but we refrain purposely from using them, as they are not needed. It is, indeed, sufficient for our proofs to know that those integrals depend only on the exponents $m, n$ and are independent of $\phi, \theta, \psi, \eta$.

University of Saskatchewan

---------

**ON THE DENSITY THEOREM**

A. PAPOULIS

1. **Introduction.** Let $F$ be a set on the plane and $x$ a point of $F$. With $\{I_n\}$ an arbitrary sequence of intervals$^1$ containing the point $x$ and with diameter tending to zero, we form the sequence $|F \cdot I_n|/|I_n|^2$. It has been shown (see [1] and [2])$^3$ that for almost$^4$ all points $x$ of $F$,

$$\lim_{I_n} \frac{|F \cdot I_n|}{|I_n|} = 1.$$  

If the sequence $\{I_n\}$ of intervals is replaced by a sequence of arbitrary rectangles with sides not necessarily parallel to the axes of coordinates, then the above ceases to be true. H. Busemann and W. Feller (see [1]) have shown that if the direction of some one of the sides of the rectangles $\{I_n\}$ varies within any nonzero angle, then (1) is no longer true for all sets $F$.

The purpose of the following is to show that even if the direction of the rectangles $\{I_n\}$ converging to the point $x$ is fixed, then (1) is still not true for some sets, provided of course that the fixed direction may vary from point to point.

Received by the editors October 17, 1950.

$^1$ Rectangles with sides parallel to the coordinate axes.

$^2$ The number $|E|$ will mean the two-dimensional Lebesgue-measure of the set $E$.

$^3$ Numbers in brackets refer to the references at the end of the paper.

$^4$ By "almost all points $x$ of a set $E$" we shall mean all points of $E$ except for a set of measure zero; this will also be indicated by p.p.