ON THE INTEGRATION SCHEME OF MARÉCHAL

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J. E. Wilkins, Jr.[1] proves the following assumption of A. Maréchal:[2] Let $f(x, y)$ be a function of 2 variables, continuous in the interior of the circle C, of radius R; then

$$\iint_C f(x, y)d\sigma = \lim_{a\to 0} \left\{2\pi a \int_{S_a} f(x, y)\cdot ds\right\}, \tag{1}$$

where the double integral extends over the area of C and the line integral is taken along the arc of the archimedean spiral (S_a)

$$(S_a) \qquad r = a\phi \tag{2}$$

interior to C.

In what follows, we give a short, elementary proof of (1), and two extensions.

I. **Proof of (1).** Let $x = r\cos\phi$, $y = r\sin\phi$ and use the notations:

$$\begin{aligned} A(m, n) &= \frac{1}{2\pi}\int_0^{2\pi} \cos^m\phi \sin^n\phi d\phi, \\ B(m, n) &= \frac{1}{\pi}\int_0^{\pi} \cos^m\phi \sin^n\phi d\phi, \\ C(m) &= \int_0^R r^m dr. \end{aligned} \tag{3}$$

Then, in (1), $d\sigma = rdrd\phi$ and $ds = (r^2+a^2)^{1/2}\cdot d\phi$.

As any continuous function can be approximated by a uniformly convergent sequence of polynomials, it is sufficient to prove (1) for $f(x, y) = x^m y^n = r^{m+n}\cos^m\phi \sin^n\phi$. If $m = n = 0$, (1) is verified by direct integration. If $m+n>0$, the first member of (1) becomes

$$\int_0^R \int_0^{2\pi} r^{m+n+1}\cos^m\phi \sin^n\phi drd\phi = 2\pi A(m, n)\cdot C(m+n+1); \tag{4}$$

and the second member may be written as

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[1] Bull. Amer. Math. Soc. vol. 55 (1949) pp. 191–192.

[2] A. Maréchal, *Mechanical integrator*, Journal of the Optical Society of America vol. 37 (1947) pp. 403–404.

$$(5)\qquad \begin{aligned} &\lim_{a\to 0}\left\{2\pi a\int_{S_a} r^{m+n}(r^2+a^2)^{1/2}\cos^m\phi\sin^n\phi d\phi\right\} \\ &\qquad = \lim_{a\to 0} 2\pi\int_0^R \left\{r^{m+n+1}\cos^m(r/a)\cdot\sin^n(r/a)+O(a)\right\}dr. \end{aligned}$$

In the (finite) Fourier expansion $\cos^m(r/a)\cdot\sin^n(r/a)=a_0/2+\sum_{\nu=1}^{m+n}(a_\nu\cos(\nu r/a)+b_\nu\sin(\nu r/a))$ the first term is $a_0/2=(1/2\pi)\int_0^{2\pi}\cos^m\phi\sin^n\phi d\phi=A(m, n)$. When $a\to 0$, then $\nu/a\to\infty$, so that, by the Riemann-Lebesgue lemma on Fourier series,

$$\lim_{a\to 0}\int_0^R r^{m+n+1}\cos(\nu r/a)dr = 0, \qquad \lim_{a\to 0}\int_0^R r^{m+n+1}\sin(\nu r/a)dr = 0,$$

$$\nu = 1, 2, \cdots, m+n,$$

and (5) reduces to $\lim_{a\to 0}\left\{2\pi\int_0^R r^{m+n+1}A(m, n)dr+2\pi R\cdot O(a)\right\}=2\pi A(m, n)\cdot C(m+n+1)$, the same as (4), proving (1).

II. Consider the integral of the continuous function $f(x, y, z)$ taken on the surface of the sphere S of radius R. We may approximate it by the integral taken on a narrow strip, winding around the sphere, along the path

$$(2')\qquad (S'_a)\qquad R\phi = a\theta,$$

from one pole ($\phi=\theta=0$) to the other ($\phi=\pi$, $\theta=R\pi/a$). We take the width of the strip to be $2\pi a$ and then make a tend to zero. The relation similar to (1) which we want to prove is, therefore,

$$(6)\qquad \iint_S f(x, y, z)d\sigma = \lim_{a\to 0}\left\{2\pi a\int_{S'_a} f(x, y, z)ds\right\}.$$

As before, it is sufficient to prove (6) for $f(x, y, z)=x^m y^n z^k$, $m+n+k>0$, because, for $m=n=k=0$, $f(x, y, z)=1$ and (6) is verified by direct integration. The first member of (6) becomes successively, using (3), $\iint_S R^{m+n+k}\sin^{m+n}\phi\cos^k\phi\cos^m\theta\sin^n\theta\cdot R^2\sin\phi d\phi d\theta=R^{m+n+k+2}\int_0^{2\pi}\cos^m\theta\sin^n\theta d\theta\int_0^\pi\sin^{m+n+1}\phi\cos^k\phi d\phi=R^{m+n+k+2}\cdot 2\pi A(m, n)\cdot\pi B(k, m+n+1)=2\pi^2R^{m+n+k+2}\cdot A(m, n)\cdot B(k, m+n+1)$. The second member of (6) becomes, as under I,

$$\begin{aligned} &\lim_{a\to 0}\left\{2\pi a\int_{S'_a} R^{m+n+k}\sin^{m+n}\phi\cos^k\phi\cos^m\theta\sin^n\theta\cdot(R\sin\phi+O(a))d\theta\right\} \\ &= \lim_{a\to 0}\left\{2\pi a\cdot R^{m+n+k+1}\int_{S'_a}\sin^{m+n+1}\phi\cos^k\phi\cos^m\theta\sin^n\theta d\theta+O(a)\right\}. \end{aligned}$$

By (2′), $\theta=\phi R/a$, so that the last expression becomes

$$(8)\quad \lim_{a\to 0}\left\{2\pi R^{m+n+k+2}\int_0^\pi \sin^{m+n+1}\phi\cos^k\phi\cos^m(R\phi/a)\cdot\sin^n(R\phi/a)d\phi\right\}.$$

Here $g(\phi)=\sin^{m+n+1}\phi\cos^k\phi$ is a continuous, bounded function and we use, as under I, the relation

$$\cos^m(R\phi/a)\cdot\sin^n(R\phi/a) = a_0/2+\sum_{\nu=1}^{m+n}(a_\nu\cos(R\phi\nu/a)+b_\nu\sin(R\phi\nu/a))$$

with $a_0/2=A(m, n)$. When $a\to 0$, $R\nu\phi/a\to\infty$ and it follows from the Riemann-Lebesgue lemma that all the expressions of the form

$$\lim_{a\to 0}\int_0^\pi g(\phi)\cos(R\nu\phi/a)d\phi \quad\text{and}\quad \lim_{a\to 0}\int_0^\pi g(\phi)\sin(R\nu\phi/a)d\phi,$$
$$\nu = 1, 2, \cdots, m+n,$$

vanish and (8) reduces to

$$2\pi R^{m+n+k+2}A(m, n)\int_0^\pi \sin^{m+n+1}\phi\cos^k\phi d\phi$$
$$= 2\pi^2R^{m+n+k+2}A(m, n)\cdot B(k, m+n+1),$$

same as (7), proving (6).

III. Let the sphere S^ν, of radius r_ν, be covered by a wire of square section $2\pi a\times 2\pi a$, winding on the sphere along a spiral like (S_a'). The outer surface of the wire is a new sphere of radius $r_{\nu+1}=r_\nu+2\pi a$ and let this be covered in the same way, by the same wire, and so forth. In particular, making $a\to 0$, we can fill the interior of the sphere S, of radius R, with such successive layers of wire, winding along spirals of equations[3]

$$(2'')\qquad (S_a^\nu)\qquad r_\nu\phi = a\theta,\qquad \nu = 1, 2, \cdots, [R/2\pi a],$$

where, in the νth layer from the center, $r_\nu=\nu(2\pi a)$. We may attempt to approximate an integral, extended over the volume of the sphere, by the sum of integrals taken along the (S_a^ν), which wind around the successive spherical shells, and are led to consider the equality

$$\iiint_V f(x, y, z)d\tau = \lim_{a\to 0}\left\{4\pi^2a^2\sum_{\nu=1}^{[R/2\pi a]}\int_{S_a^\nu} f(x, y, z)\cdot ds\right\},$$

where the integral in the first member is extended over the volume of

[3] $[R/2\pi a]$ stands for the largest integer not exceeding $R/2\pi a$.

S and the integrals of the second member are taken along the arcs $s^{(\nu)}$ of the spirals (S_a^ν) from (2″). The proof, proceeding along the same lines as that of (6), is suppressed here.

REMARK. The explicit values of the elementary integrals (3) are, of course, well known; but we refrain purposely from using them, as they are not needed. It is, indeed, sufficient for our proofs to know that those integrals depend only on the exponents m, n and are independent of ϕ, θ, or r.

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ON THE DENSITY THEOREM

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1. **Introduction.** Let F be a set on the plane and x a point of F. With $\{I_n\}$ an arbitrary sequence of intervals[1] containing the point x and with diameter tending to zero, we form the sequence $|F\cdot I_n|/|I_n|$.[2] It has been shown (see [1] and [2])[3] that for almost[4] all points x of F,

$$\lim_{I_n} \frac{|F\cdot I_n|}{|I_n|} = 1. \tag{1}$$

If the sequence $\{I_n\}$ of intervals is replaced by a sequence of arbitrary rectangles with sides not necessarily parallel to the axes of coordinates, then the above ceases to be true. H. Busemann and W. Feller (see [1]) have shown that if the direction of some one of the sides of the rectangles $\{I_n\}$ varies within any nonzero angle, then (1) is no longer true for all sets F.

The purpose of the following is to show that even if the direction of the rectangles $\{I_n\}$ converging to the point x is fixed, then (1) is still not true for some sets, provided of course that the fixed direction may vary from point to point.

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[1] Rectangles with sides parallel to the coordinate axes.

[2] The number $|E|$ will mean the two-dimensional Lebesgue-measure of the set E.

[3] Numbers in brackets refer to the references at the end of the paper.

[4] By "almost all points x of a set E" we shall mean all points of E except for a set of measure zero; this will also be indicated by p.p.