Further $\mathfrak{F}$ can, apart from trivial (unit) factors, be expressed in at most one way as a product of indecomposable factors. The same results hold for families of regular bilinear mappings ($n=2$, $H=K$, $H_i=K_i$) and for families of groups of class 1 or 2. If either Q or G/Z be finite, $F(G)$ is uniquely expressible as a product of indecomposable families.

Reference

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A SHORT PROOF OF AN IDENTITY OF EULER

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Euler discovered the identity

$$\prod_{s=1}^{\infty}(1-x^s) = 1 + \sum_{s=1}^{\infty}(-1)^s[x^{s(3s-1)/2} + x^{s(3s+1)/2}]. \tag{1}$$

He used it in the theory of partitions, and, after some time, he proved it [1].[1] Later, famous proofs involving theta functions and combinatorial arguments were given by Jacobi and F. Franklin [2]. The following algebraic proof is quite simple.

Let the partial products and partial sums of (1) be

$$P_0 = 1, \qquad P_n = \prod_{s=1}^{n}(1-x^s),$$

and

$$S_n = 1 + \sum_{s=1}^{n}(-1)^s[x^{s(3s-1)/2} + x^{s(3s+1)/2}].$$

Then S_n and P_n are related by the *finite* identity

$$S_n = F_n \qquad \text{where } F_n = \sum_{s=0}^{n}(-1)^s \frac{P_n}{P_s} x^{sn+s(s+1)/2}. \tag{2}$$

To prove (2) we detach the last term, $s=n$, and split the remaining sum into two parts by putting $P_n = P_{n-1} - x^n P_{n-1}$. This gives

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[1] Numbers in brackets refer to the references cited at the end of the paper.

$$F_n = \sum_{s=0}^{n-1} (-1)^s \frac{P_{n-1}}{P_s} x^{sn+s(s+1)/2} + \sum_{r=0}^{n-1} (-1)^{r+1} x^n \frac{P_{n-1}}{P_r} x^{rn+r(r+1)/2}$$
$$+ (-1)^n x^{n(3n+1)/2}.$$

We detach the term $r=n-1$, and recombine the two sums by putting $r=s-1$ and adding the new corresponding terms. By using $1/P_{s-1} = (1-x^s)/P_s$ we obtain

$$F_n = \sum_{s=0}^{n-1} (-1)^s \frac{P_{n-1}}{P_s} x^{s(n-1)+s(s+1)/2} + (-1)^n \left[x^{n(3n-1)/2} + x^{n(3n+1)/2}\right];$$

or, referring to the definitions of F_n and S_n,

$$F_n = F_{n-1} + (S_n - S_{n-1}).$$

Therefore $S_n = F_n$ if $S_{n-1} = F_{n-1}$ and since

$$S_1 = 1 - (x + x^2) = (1 - x) - x^2 = F_1,$$

equation (2) follows by induction.

Now the partial product, P_n, is the first term of F_n ($s=0$ in (2)), and since the remaining terms of F_n are of order x^{n+1} and higher, P_n must agree with F_n up to x^n. Therefore P_n agrees with S_n up to x^n and, letting $n \to \infty$, we see that both sides of (1) have the same power series. This proves the theorem.

The origin of the formula, F_n, is of interest. It was obtained by applying a nonlinear transformation [3] to the sequence of products, $A_i = 1/P_i$. Given the sequence A_i ($i=0, 1, \cdots, 2k$), one obtains the transform, B_{kk}, from the formula:

$$B_{kk} = \frac{\begin{vmatrix} A_0 & A_1 & A_2 & \cdots & A_k \\ \Delta A_0 & \Delta A_1 & \cdots & \cdots & \Delta A_k \\ \Delta A_1 & \Delta A_2 & \cdots & \cdots & \Delta A_{k+1} \\ \vdots & & & & \vdots \\ \Delta A_{k-1} & \cdots & \cdots & \cdots & \Delta A_{2k-1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \Delta A_0 & \Delta A_1 & \cdots & \cdots & \Delta A_k \\ \Delta A_1 & \Delta A_2 & \cdots & \cdots & \Delta A_{k+1} \\ \vdots & & & & \vdots \\ \Delta A_{k-1} & \cdots & \cdots & \cdots & \Delta A_{2k-1} \end{vmatrix}} \quad \text{where } \Delta A_i = A_{i+1} - A_i.$$

Reduction of the determinants gives $B_{kk} = 1/F_k$. The members of the original sequence, A_0 to A_{2k}, agree with the infinite series to, at most,

x^{2k}, and since the transform B_{kk} agrees to $x^{k(3k+5)/2}$, we see that the transformation greatly increases the rate of convergence of the sequence.

By a very similar calculation (proof omitted), one finds that if

$$Q_0 = 1, \ Q_n = \prod_{s=1}^{n} \frac{(1 - x^{2s})}{(1 - x^{2s-1})}, \text{ and } T_n = 1 + \sum_{s=1}^{n} [x^{s(2s-1)} + x^{s(2s+1)}],$$

then $T_n = G_n$ where $G_n = \sum_{s=0}^{n} (Q_n/Q_s)x^{s(2n+1)}$. This implies $Q_\infty = T_\infty$, an identity due to Gauss [2].

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