FAMILIES OF LORENTZIAN MATRICES

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1. **The differential equation.** Let $P(s)$ be an n-rowed square matrix whose elements are continuous real functions of the real variable s in an interval $s_0 \leqq s \leqq s_1$. The differential equation

$$\Lambda'(s) = P(s)\cdot\Lambda(s), \qquad \Lambda(s) = (\lambda_{ij}),$$

is equivalent to the n systems of equations

$$\begin{aligned}
\lambda'_{1i} &= p_{11}\lambda_{1i} + p_{12}\lambda_{2i} + \cdots + p_{1n}\lambda_{ni},\\
\lambda'_{2i} &= p_{21}\lambda_{1i} + p_{22}\lambda_{2i} + \cdots + p_{2n}\lambda_{ni},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
\lambda'_{ni} &= p_{n1}\lambda_{1i} + p_{n2}\lambda_{2i} + \cdots + p_{nn}\lambda_{ni}, \qquad i = 1, 2, \cdots, n.
\end{aligned} \tag{1}$$

The coefficients are independent of i so that each column of $\Lambda(s)$ is a solution of this system.

These equations are known [1; 2; 3][1] to have a solution for every choice of initial conditions, and to have exactly n linearly independent solutions, every solution then being a linear combination of any set of n linearly independent ones. This is equivalent to the statement in matric notation that there is a solution $\Lambda(s)$ such that $|\Lambda(s_0)| \neq 0$, and that every solution is then given by $\Lambda(s)\cdot A$ where A is an arbitrary constant real matrix. If the elements of $P(s)$ are $n-1$ times differentiable, it can be shown that each element of $\Lambda(s)$ satisfies a linear differential equation whose coefficients are polynomials in the elements of $P(s)$ and their derivatives.

If (1) has a solution $\Lambda(s)$ all of whose elements are analytic functions of s in a neighborhood of $s=0$, it may be expressed in power series. From (1) we have

$$\begin{aligned}
\Lambda'(0) &= P(0)\cdot\Lambda(0),\\
\Lambda''(s) &= P'(s)\cdot\Lambda(s) + P(s)\cdot\Lambda'(s) = [P'(s) + P^2(s)]\Lambda(s).
\end{aligned}$$

Similarly

$$\Lambda^{(i)}(s) = M_i(s)\cdot\Lambda(s)$$

where $M_i(s)$ is a polynomial in $P(s)$ and its derivatives. Then near

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[1] Numbers in brackets refer to the references at the end of the paper.

$s=0$ [4; 5]

$$\Lambda(s) = \left[I + M_1(0)s + \frac{1}{2} M_2(0)s^2 + \frac{1}{3!} M_3(0)s^3 + \cdots \right]\Lambda(0).$$

In the case where P is a constant matrix, this becomes particularly simple, for $\Lambda^{(i)}(s) = P^i\Lambda(s)$ so that

$$\begin{aligned}\Lambda(s) &= \left[I + P\cdot s + \frac{1}{2} P^2\cdot s^2 + \frac{1}{3!} P^3\cdot s^3 + \cdots \right]\Lambda(0) \\ &= e^{P\cdot s}\Lambda(0),\end{aligned} \tag{2}$$

which is a known result. In fact, Hurewicz [3, p. 43] has shown that $\Lambda(s) = e^{\int P(s)\cdot ds}\Lambda(0)$ whenever $P(s)$ is commutative with its integral.

All the results of this section may be paralleled for the equation

$$\Lambda'(s) = \Lambda(s)\cdot Q(s)$$

by taking the transposed matrices.

2. **Lorentzian matrices.** We shall denote by J an n by n matrix whose elements are real numbers and which is both symmetric and orthogonal. Since J is symmetric, there exists a real orthogonal matrix O such that

$$O^TJO = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = D$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the characteristic roots of J, and are all real. Since J is orthogonal, these roots are of absolute value unity so that $\lambda_i = \pm 1$. Clearly

$$J = J^T = J^{-1}, \qquad |J| = \pm 1.$$

The set of all matrices A such that

$$A^TJA = J$$

constitute a group which we shall call a *Lorentzian group* and denote by $\mathfrak{L}_J$. It is merely a matter of an isomorphism

$$A \leftrightarrow O^TAO$$

to suppose that J is actually the diagonal matrix D. If $D = I$, then $\mathfrak{L}_I$ is the orthogonal group of n by n matrices so that the theory of orthogonal matrices is contained in this treatment.

THEOREM 1. *If $A \in \mathfrak{L}_J$, then also $A^T \in \mathfrak{L}_J$.*

For if $A^TJA = J$, then $A^TJAJ = J^2 = I$ so that

$$AJ = (A^TJ)^{-1} = J^{-1}A^{-T} = JA^{-T}.$$

That is, $AJA^T = J$.

THEOREM 2. *Let $\Lambda(s)$ be a matrix whose elements are differentiable real functions of s which is in $\mathfrak{L}_J$ for every s in the interval $s_0 \leqq s \leqq s_1$. Then there exist unique matrices $P(s)$ and $Q(s)$ such that $J \cdot P(s)$ and $Q(s) \cdot J$ are skew, and*

$$\Lambda'(s) = P(s) \cdot \Lambda(s) = \Lambda(s) \cdot Q(s).$$

Upon differentiating $\Lambda^T(s) \cdot J \cdot \Lambda(s) = J$, we have

$$\Lambda'^T(s) \cdot J \cdot \Lambda(s) + \Lambda^T(s) \cdot J \cdot \Lambda'(s) = 0.$$

That is,

$$\Lambda^T(s) \cdot P^T(s) \cdot J \cdot \Lambda(s) + \Lambda^T(s) \cdot J \cdot P(s) \cdot \Lambda(s) = 0,$$
$$\Lambda^T(s)[P^T(s) \cdot J + J \cdot P(s)]\Lambda(s) = 0.$$

Since $|\Lambda(s)| = \pm 1$, $\Lambda(s)$ is nonsingular and

$$J \cdot P(s) = -P^T(s) \cdot J = -(J \cdot P(s))^T$$

so that $J \cdot P(s)$ is skew.

Upon differentiating $\Lambda(s) \cdot J \cdot \Lambda^T(s) = J$, we obtain similarly

$$\Lambda'(s) = \Lambda(s) \cdot Q(s)$$

where $Q(s) \cdot J$ is skew.

Clearly $P(s) \cdot J$ is skew if and only if $J \cdot P(s)$ is skew, and similarly $Q(s) \cdot J$ is skew if and only if $J \cdot Q(s)$ is skew.

Since for every s in the interval $s_0 \leqq s \leqq s_1$, $|\Lambda(s)| = \pm 1$, it follows that $\Lambda'(s) \cdot \Lambda^{-1}(s) = P(s)$ so that $P(s)$ is unique. Similarly $Q(s) = \Lambda^{-1}(s) \cdot \Lambda'(s)$ is unique.

THEOREM 3. *Let $P(s)$ be a matrix whose elements are continuous real functions of s in the interval $s_0 \leqq s \leqq s_1$ having the property that $P(s) \cdot J$ is skew. Let $\Lambda(s)$ be a matrix which satisfies a differential equation*

$$\Lambda'(s) = P(s) \cdot \Lambda(s),$$

subject to the initial condition that $\Lambda(s_2) \in \mathfrak{L}_J$ for some value s_2 of s in the interval $s_0 \leqq s \leqq s_1$. Then $\Lambda(s) \in \mathfrak{L}_J$ throughout the interval.

Set $\Lambda^T(s) \cdot J \cdot \Lambda(s) = R(s)$. Then as before

$$\Lambda^T(s)[P^T(s)\cdot J + J\cdot P(s)]\Lambda(s) = R'(s).$$

Since $J\cdot P(s)$ is skew, $R'(s)=0$ and $R(s)$ is a constant matrix. That is, $R(s)$ has for every value of s in the interval the value that it has for $s=s_2$, namely J. Thus $\Lambda(s)\in\mathfrak{L}_J$.

References

1. C. de la Vallée Poussin, *Cours d'analyse infinitésimale*, Louvin, 1937, p. 259.

2. S. Lefschetz, *Lectures on differential equations*, Princeton University Press, 1946, chap. III.

3. W. Hurewicz, *Ordinary differential equations in the real domain with emphasis on geometric methods*, Providence, 1943, p. 39.

4. G. Sansone, *Equazioni differenziali nel campo reale*, part 1, 2d ed., Bologna, 1948, p. 131.

5. R. A. Frazer, W. J. Duncan, and A. R. Collar, *Elementary matrices*, Cambridge, 1938, p. 222.

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