S and the integrals of the second member are taken along the arcs $s^{(n)}$ of the spirals $(S_n)$ from $(2^n)$. The proof, proceeding along the same lines as that of (6), is suppressed here.

Remark. The explicit values of the elementary integrals (3) are, of course, well known; but we refrain purposely from using them, as they are not needed. It is, indeed, sufficient for our proofs to know that those integrals depend only on the exponents $m$, $n$ and are independent of $\phi$, $\theta$, or $r$.

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ON THE DENSITY THEOREM

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1. Introduction. Let $F$ be a set on the plane and $x$ a point of $F$. With $\{I_n\}$ an arbitrary sequence of intervals$^1$ containing the point $x$ and with diameter tending to zero, we form the sequence $|F \cdot I_n|/|I_n|^2$. It has been shown (see [1] and [2])$^3$ that for almost$^4$ all points $x$ of $F$,

$$\lim_{I_n} \frac{|F \cdot I_n|}{|I_n|} = 1.$$  

(1)

If the sequence $\{I_n\}$ of intervals is replaced by a sequence of arbitrary rectangles with sides not necessarily parallel to the axes of coordinates, then the above ceases to be true. H. Busemann and W. Feller (see [1]) have shown that if the direction of some one of the sides of the rectangles $\{I_n\}$ varies within any nonzero angle, then (1) is no longer true for all sets $F$.

The purpose of the following is to show that even if the direction of the rectangles $\{I_n\}$ converging to the point $x$ is fixed, then (1) is still not true for some sets, provided of course that the fixed direction may vary from point to point.

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$^1$ Rectangles with sides parallel to the coordinate axes.

$^2$ The number $|E|$ will mean the two-dimensional Lebesgue-measure of the set $E$.

$^3$ Numbers in brackets refer to the references at the end of the paper.

$^4$ By "almost all points $x$ of a set $E$" we shall mean all points of $E$ except for a set of measure zero; this will also be indicated by p.p.
2. Part A. We shall construct⁶ a set $E$ included in the unit square $Q$ and such that:

(2A) $|E| < 3\varepsilon$, where $\varepsilon$ is an arbitrary number; given $x \in Q$ there exists a sequence $\{I_n\}$ of rectangles containing the point $x$, with diameter tending to zero, and such that:

$$\limsup_{I_n} \frac{|E \cdot I_n|}{|I_n|} > \frac{1}{3}.$$ 

This construction will be given in §§3–7.

3. Definition. Given a plane set $E$, let $M(E)$ be the set of points $x$ of the plane with the property that there exists a line-segment $s$ containing $x$ and such that

$$\frac{m_1(E \cdot s)}{m_1(s)} \geq \frac{1}{2}.$$ 

If $E$ is a circle of radius $R$, then clearly $M(E)$ is a concentric circle of radius $3R$.

4. Lemma.⁷

$$\text{g.l.b.} \frac{|E|}{|M(E)|} = 0$$

as $E$ ranges over all plane sets.

If $E$ is a circle, then $|E|/|M(E)| = 1/9$; if $E$ is a triangle, then $|E|/|M(E)| = 1/13$.

PROOF. It suffices to show that given an $\varepsilon$ we can construct a set $E$ such that

$$|E|/|M(E)| < \varepsilon.$$ 

Consider the open isosceles triangle $S_0 = ABC$ of base $BC = a$, angle $BAC = \theta$, and altitude $h$, with an axis $\alpha \parallel \alpha$ parallel to the $y$-axis. Clearly the points of the trapezoid $T_0 = BCED$, where $DE$ is the line $y = -h$, belong to $M(S_0)$. Draw the lines $y = h - h'$ and $y = h + h'$

⁶ This construction is similar to the one given by H. Busemann and W. Feller in [1].

⁷ The number $m_1(s)$ will mean the linear measure of the set $s$.

⁷ The above lemma is based on a construction used by O. Perron in giving a simple solution to the Besicovitch-Kakeya problem; it was utilized also by H. Busemann and W. Feller in their treatment of the density theorem.
(h' to be determined soon); they intersect AB and AC and their extensions at the points K, L and L', K'; the lines KK' and LL' intersect BC at the points M and N. The open figure $S_1 = AK'KBCLL'A$ has an area $|S_1| = |S_0| \{1 + 2(h'/h)^2\}$; the line $y = -(h+h')$ intersects the extensions of AB and AC at the points $D_1$ and $E_1$ forming the trapezoid $T_1 = BCE_1D_1$; clearly $|T_1| > |T_0|(1+h'/h)$ and $T_1 \subset M(S_1)$.

With $h' = h/2$ we have

$$|S_1| = |S_0| \{1 + 2(1/2)^2\}, \quad |T_1| > |T_0| \{1 + 1/2\}.$$

We repeat the above construction with each of the triangles $K'MC$ and $L'BN$ with $h' = h/3$; we thus obtain the open figure $S_3$ consisting of $S_1$ and the four added triangles II, and the trapezoid $T_3 = BCE_2D_2$. Again
Repeating the above process on each of the four new triangles, whose base is part of $BC$, with $h' = h/4$, we obtain the open figure $S_3$, consisting of $S_2$ and the eight added triangles, and the trapezoid $T_3 = BCE_3D_3$; as before

$$T_2 \subset M(S_2),$$
$$|S_2| = |S_0| \{1 + 2(1/2)^2 + 2(1/3)^2\},$$
$$|T_2| > |T_0| \{1 + 1/2 + 1/3\}.$$

At the $n$th such step with $h' = h/n$, we obtain the open figure $S_n$ and the trapezoid $T_n = BCE_nD_n$; they satisfy the relation

$$T_n \subset M(S_n),$$
$$|S_n| = |S_0| \{1 + 2(1/2)^2 + \cdots + 2(1/n)^2\},$$
$$|T_n| > |T_0| \{1 + 1/2 + \cdots + 1/n\}.$$

Since $1 + 1/2 + \cdots + 1/n + \cdots$ diverges, $1 + (1/2)^2 + \cdots + (1/n)^2 + \cdots$ converges, and $|T_0| \neq 0$, it follows that for a certain $n = N(\varepsilon)$ we shall have

$$|S_n| < \varepsilon,$$

and since $S_n + T_N \subset M(S_N)$, (4) has been established and the lemma is proved.

5. The set $S_N + T_N$ will be denoted by $B(\varepsilon; \theta)$, where $\varepsilon$ and $\theta$ are as in the previous section, and the set $S_N$ by $S(\varepsilon; \theta)$. Thus

$$\frac{|S_N|}{|S_N + T_N|} < \varepsilon,$$

and since $S_N + T_N \subset M(S_N)$, (4) has been established and the lemma is proved.

6. Construction of $E$. In this section $\theta$ will be kept constant and will be omitted in the expressions $S(\varepsilon; \theta)$ and $B(\varepsilon; \theta)$.

First step. By $[B(\varepsilon)]$ we shall mean the class of all sets similar to $B(\varepsilon)$ and similarly placed, with diameter smaller than one. For every $x \in Q$, there exists a sequence of sets belonging to $[B(\varepsilon)]$, containing the point $x$ and with diameter tending to zero, hence (Vitali covering
there exists a sequence \( \{B(e)\} \) of disjoint sets belonging to [\( B(e) \)] and covering \( Q \) p.p. The sets \( S(e) \) which are parts of \( B(e) \) of the sequence \( \{B(e)\} \) form another sequence \( \{S(e)\} \) whose sum we denote by \( E_0 \); from (8) and the disjointness of the sets \( B(e) \) of \( \{B(e)\} \) we conclude that

\[
|E_0| < \epsilon |Q| = \epsilon.
\]

**Second step.** By \( [B(e/2)] \) we shall mean the class of all sets similar to \( B(e/2) \) and similarly placed with diameter less than 1/2. As before there exists a sequence \( \{B(e/2)\} \) of disjoint sets belonging to \( [B(e/2)] \) and covering \( Q \) p.p.; the sets \( S(e/2) \) which are parts of \( B(e/2) \) of the sequence \( \{B(e/2)\} \) form another sequence \( \{S(e/2)\} \) whose sum we denote by \( E_1 \); as before,

\[
|E_1| < \frac{\epsilon}{2} |Q| = \epsilon/2^k.
\]

(k+1)th step. By \( [B(e/2^k)] \) we shall mean the class of all sets similar to \( B(e/2^k) \) and similarly placed, with diameter smaller than \( 2/2^k \). Again there exists a sequence \( \{B(e/2^k)\} \) of disjoint sets belonging to \( [B(e/2^k)] \) and covering \( Q \) p.p., and the sequence \( \{S(e/2^k)\} \) of the \( S(e/2^k) \) sets whose sum we denote by \( E_k \); as before

\[
|E_k| < \epsilon/2^k
\]

and so we continue.

Since we repeat the process an enumerable number of times, the set of points of \( Q \) which are not covered at least once in this process has a zero measure; hence it can be covered by an open set \( G \) of area smaller than \( \epsilon \); with \( E = G + \sum_{k=0}^{\infty} E_k \), we have

\[
|E| \leq |G| + \sum_{k=0}^{\infty} |E_k| < \epsilon + \sum_{k=0}^{\infty} \frac{\epsilon}{2^k} = 3\epsilon.
\]

7. We shall now show that \( E \) satisfies the requirements of §2. Suppose \( x \in Q \); if \( x \) belongs to \( E \), \( E \) being open, we can cover \( x \) with a circle \( C_x \subset E \). If we take any sequence \( \{I_n\} \) of intervals included in \( C_x \) and with diameter tending to zero, we have \( |E \cdot I_n| / |I_n| = 1 > 1/3 \); hence (2A) is true. Suppose \( x \) is not a point of \( E \); it suffices to show that given \( \delta > 0 \) there exists an interval \( I_k \) of diameter smaller than \( \delta \) and such that

\[
x \in I_k, \quad \frac{|E \cdot I_k|}{|I_k|} > \frac{1}{3}.
\]
Take \( k \) such that \( 1/2^k < \delta \); since \( x \in E \) it follows that \( x \in G \), hence it is covered by one set \( B(\epsilon/2^k) \) belonging to the sequence \( \{ B(\epsilon/2^k) \} \) of the \((k+1)\)th step. From the way these sets have been constructed, it follows that there exists a line segment \( s \) such that

\[
x \in s, \quad s \subseteq B\left(\frac{\epsilon}{2^k}\right), \quad \frac{m_1(s \cdot S(\epsilon/2^k))}{m_1(s)} > \frac{1}{2}
\]

where \( S(\epsilon/2^k) \) is the set contained in \( B(\epsilon/2^k) \). Since the maximum diameter of the sets \( \{ B(\epsilon/2^k) \} \) is smaller than \( 1/2^k \), and \( s \subseteq B(\epsilon/2^k) \), we conclude that \( ds < 1/2^k \). But \( S(\epsilon/2^k) \subseteq E_k \subseteq E \), hence also

\[
\frac{m_1(s \cdot E)}{m_1(s)} > \frac{1}{2}
\]

and since \( E \) is open, we can find an interval \( I_k \) containing \( s \) and satisfying (15); and so our contention is proved.

**Remark 2.** Since the orientation of the \( I_k \)'s can be taken within the angle \( \theta \), and since \( \theta \) can be chosen arbitrarily small, it follows that the rectangles of the sequence \( \{ I_k \} \) in (2A) can be chosen with an orientation within an arbitrarily small angle.

8. With \( F = Q - E \) we have from (2A) with \( x \in F \subseteq Q \)

\[
\lim \inf \frac{|F \cdot I_n|}{|I_n|} < \frac{2}{3}.
\]

Hence (1) is not true because now we have a sequence \( \{ I_n \} \) of rectangles and not of intervals.

9. **Part B.** We shall construct a set \( E \) included in the unit square \( Q \) and such that:

\[
|E| < 5\epsilon \quad \text{where} \quad \epsilon \quad \text{is an arbitrary number}; \quad \text{given} \quad x \in Q \quad \text{there exists a sequence} \quad \{ I_n \} \quad \text{of rectangles of "fixed orientation," containing the point} \quad x, \quad \text{with diameter tending to zero, and such that}
\]

\[
\lim \sup \frac{|E \cdot I_n|}{|I_n|} > \frac{1}{3}.
\]

10. **Notations.** Suppose angle \( BAC = \theta_1 \), angle \( EDF = \theta_2 \), and \( \gamma \) a given direction; if from \( A \) we draw a line parallel to \( \gamma \) and it falls inside \( BAC \), this will be indicated by writing \( \gamma \subseteq \theta_1 \). If from \( A \) we draw two lines parallel to \( DE \) and \( DF \) respectively and both fall inside \( BAC \), this will be indicated by writing \( \theta_2 \subseteq \theta_1 \). By \( s_\gamma \) we shall mean a line segment parallel to the direction \( \gamma \).

* \( ds \) will mean the diameter of the set \( s \).
11. Outline of the method. In the construction of Part A the direction of the line segments satisfying (14), and hence of the corresponding rectangles $I_k$, cannot be the same for every $k$ for the following reason: for a given $x \in \mathcal{Q}$ the segments $s$ satisfying (14) for $k = 0$ may have any direction $\gamma \subseteq \delta^0$; for $k = 1$ may have any direction $\gamma \subseteq \theta^1$, for any $k$ a direction $\gamma \subseteq \delta^k$; the angle $\delta^k$ depends on the location of $x$ inside $B(e/2^k; \theta)$. It is possible that there is no direction $\gamma \subseteq \delta^k$ for every $k$. If we modify our second covering in such a manner that the covering sets $B(e/2; \theta')$ have an angle $\theta' \subseteq \delta^0$, then all the directions which satisfy (14) for $k = 1$ must be included in $\delta^0$, hence in $\delta$, they therefore must satisfy (14) also for $k = 0$; these possible directions form a nonzero angle $\theta^0$ which is used for the following step. Thus for each point $x$ we have a sequence of closed angles $\theta_0 \supseteq \theta_1 \supseteq \theta_2 \supseteq \cdots$, hence there exists a $\gamma \subseteq \theta^k$ and a segment $s$, satisfying (14) for every $k$; and that is our objective.

12. We now come to our construction.

First step. We begin as in §6; we thus have the sequences $\{B(e; \theta)\}$, $\{S(e; \theta)\}$ and the set $\mathcal{E}_1$ with

\begin{equation}
|\mathcal{E}_1| < \epsilon.
\end{equation}

For almost every $x \in \mathcal{Q}$ there exists an $s_{\gamma_0}$ of direction $\gamma_0$ and such that

\begin{equation}
x \in s, \quad ds < 1, \quad \frac{m_1(s \cdot \mathcal{E}_1)}{m_1(s)} > \frac{1}{2}.
\end{equation}

Since $\mathcal{E}_1$ is open, it follows that all the possible directions $\gamma$ obtained by continuous rotation of $\gamma_0$, and for which the above is true, form a nonzero angle $w(x)$ depending on $x$. Consider the points $x$ of $\mathcal{Q}$ such that $w(x) > \alpha$ where $\alpha$ is a given angle; they form a set $P_\alpha$; since to almost every $x$ of $\mathcal{Q}$ there corresponds a nonzero angle $w(x)$, $\lim P_\alpha$ as $\alpha$ tends to zero must include almost all points of $\mathcal{Q}$. Therefore given an $\epsilon$ there exists an $\alpha_1$ such that

\begin{equation}
|\mathcal{Q} - P_{\alpha_1}| < \epsilon.
\end{equation}

The points of $\mathcal{Q} - P_{\alpha_1}$ can be included in an open set $G_0$ of measure less than $\epsilon$; with $R_1 = E_1 + G_1$ we have

\begin{equation}
|R_1| < \epsilon + \epsilon = 2\epsilon.
\end{equation}

Consider all points $x$ of $\mathcal{Q}$ such that for every $\gamma \subseteq \theta$ there exists an $s$ such that (16) is true; they form a set which we denote by $1A_{\theta_0}$; clearly all points of $E_1$ belong to $1A_{\theta_0}$ since their corresponding $w$ equals $2\pi$. 

In the following, angles will be considered with their orientation; thus when we speak of the set \( B(\varepsilon; \phi) \) we shall mean a set as in §5 but with its axis \( \alpha - \alpha \) not parallel to the \( y \)-axis but to the bisector of the angle \( \phi \).

If we bisect \( \theta \), we obtain the angles \( \theta^1 \) and \( \theta^2 \). We denote by \( 1A_1^1 \) \((t=1, 2)\) the set of points of \( Q - 1A_1^1 \) such that for every \( \gamma \subset \theta^1 \) \((t=1, 2)\) there exists an \( s_\gamma \) satisfying (16). We continue as follows:

We divide \( \theta \) into \( 2^n \) \((n=2, 3, \ldots)\) equal angles \( \theta^t_n \) \((t=1, 2, \ldots, 2^n)\) and denote by \( 1A_n^t \) \((t=1, \ldots, 2^n)\) the set of points of \( Q - \{1A_1^1 + 1A_1^2 + \cdots + 1A_{n-1}^1 + \cdots + 1A_{n-1}^{2^n-1}\} \) with the property that for every \( \gamma \subset \theta_n^t \) there exists an \( s_\gamma \) satisfying (16). After \( n_1 \) such steps, with \( \theta/2^{n_1+1} > \alpha_1 \), all points of \( Q - G_0 \) will be included in the sets \( 1A_h^t \) \((h=0, 1, \ldots, n_1; t=1, 2, \ldots, 2^n)\) (see (17)).

To avoid three indices let us enumerate the sets \( 1A_n^t \) and their corresponding angles \( \theta_n^t \) writing \( 1A(i) \) and \( \theta(i) \) with \( i=1, 2, \ldots, N_i \).

\( k \)th step. Assuming that we have defined \((k-1)A(i), (k-1)\theta(i), R_{k-1} \) and \( E_{k-1} \), we define \( kA(i), k\theta(i), R_k \), and \( E_k \) as follows: we cover p.p. the open kernel of \((k-1)A(i)\) (which has the same measure as \((k-1)A(i)\) as we can easily see) with a sequence \( \{B(\varepsilon/2^{k-1}; (k-1)\theta(i))\} \) of disjoint sets similar to \( B(\varepsilon/2^{k-1}; (k-1)\theta(i)) \) and similarly placed. The sets \( S \) which are parts of the \( B \)'s form another sequence \( \{S(\varepsilon/2^{k-1}; (k-1)\theta(i))\} \) whose sum we denote by \( E_k \); clearly

\[
|E_k| < \frac{\varepsilon}{2^{k-1}}.
\]

The subdivision of \( Q \) in the first step into the sets \( 1A_n^t \) we now perform on \((k-1)A(i)\) for every \( i \), the only change being that instead of (16) we now must satisfy

\[
x \in s, \quad ds < \frac{1}{2^{k-1}}, \quad \frac{m_1(s \cdot E_{k-1})}{m_1(s)} > \frac{1}{2}.
\]

We thus obtain the sets \( kA(i) \) and the angles \( k\theta(i) \); we continue this process until we cover all points of the \((k-1)A(i)\) except for a set of measure less than \( \varepsilon/2^{k-1} \) (see (17)) which we include in an open set \( G_k \) of measure less than \( \varepsilon/2^{k-1} \); and with \( R_k = E_k + G_k \) we have

\[
|R_k| < \frac{\varepsilon}{2^{k-1}} + \frac{\varepsilon}{2^{k-1}} = \frac{\varepsilon}{2^{k-2}}.
\]
The sets $kA(i)^t$ and $k\theta(i)^t$ we again enumerate for all $i$, $h$, and $t$, and we thus obtain $kA(i)$ and $k\theta(i)$.

We continue similarly. In this process the part of $Q$ which is not covered at least once by the sets $\{B\}$ has a zero measure; we cover it with a set $G$ of area smaller than $\epsilon$; with $E = G + \sum_{i=1}^{t} R_k$, we have $|E| < \epsilon + \sum_{i=1}^{t} \epsilon/2^{k-2} = 5\epsilon$.

14. We shall now prove that $E$ satisfies (2B). Suppose $x \in Q$ if $x \in G_k + G$; then we have no problem since $x$ can be covered by a circle included in $E$.

Suppose then that $x \in G_k + G$; then $x \in B(\epsilon/2^{k-1}; (k-1)\theta(i))$ for a certain $i$ depending on $x$ (and $k$), and every $k$. From the way the different coverings have been performed, it follows that

$$1\theta(i) \supset 2\theta(i) \supset \cdots \supset (k-1)\theta(i) \supset \cdots$$

and since these angles are closed there must exist a direction $\gamma_x$ included in all of them and such that (20) is true for every $k$ if the segment $s$ is parallel to $\gamma_x$; from that and the openness of $E$, (2B) follows easily.

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