S and the integrals of the second member are taken along the arcs $s_{\mu}$ of the spirals $(S_{\mu})$ from $(2^\mu)$. The proof, proceeding along the same lines as that of (6), is suppressed here.

**Remark.** The explicit values of the elementary integrals (3) are, of course, well known; but we refrain purposely from using them, as they are not needed. It is, indeed, sufficient for our proofs to know that those integrals depend only on the exponents $m, n$ and are independent of $\phi, \theta$, or $r$.

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**ON THE DENSITY THEOREM**

A. PAPOULIS

1. *Introduction.* Let $F$ be a set on the plane and $x$ a point of $F$. With $\{I_n\}$ an arbitrary sequence of intervals\(^1\) containing the point $x$ and with diameter tending to zero, we form the sequence $|F \cdot I_n| / |I_n|$.\(^2\) It has been shown (see [1] and [2])\(^3\) that for almost\(^4\) all points $x$ of $F$,

$$\lim_{I_n} \frac{|F \cdot I_n|}{|I_n|} = 1.$$  

If the sequence $\{I_n\}$ of intervals is replaced by a sequence of arbitrary rectangles with sides not necessarily parallel to the axes of coordinates, then the above ceases to be true. H. Busemann and W. Feller (see [1]) have shown that if the direction of some one of the sides of the rectangles $\{I_n\}$ varies within any nonzero angle, then (1) is no longer true for all sets $F$.

The purpose of the following is to show that even if the direction of the rectangles $\{I_n\}$ converging to the point $x$ is fixed, then (1) is still not true for some sets, provided of course that the fixed direction may vary from point to point.

\(^1\) Rectangles with sides parallel to the coordinate axes.
\(^2\) The number $|E|$ will mean the two-dimensional Lebesgue-measure of the set $E$.
\(^3\) Numbers in brackets refer to the references at the end of the paper.
\(^4\) By "almost all points $x$ of a set $E$" we shall mean all points of $E$ except for a set of measure zero; this will also be indicated by p.p.
2. Part A. We shall construct a set \( E \) included in the unit square \( Q \) and such that:

\[
\tag{2A} |E| < 3\varepsilon, \text{ where } \varepsilon \text{ is an arbitrary number; given } x \in Q \text{ there exists a sequence } \{I_n\} \text{ of rectangles containing the point } x, \text{ with diameter tending to zero, and such that:}
\]

\[
\lim \sup_{I_n} \frac{|E \cap I_n|}{|I_n|} > \frac{1}{3}.
\]

This construction will be given in §§3–7.

3. Definition. Given a plane set \( E \), let \( M(E) \) be the set of points \( x \) of the plane with the property that there exists a line-segment \( s \) containing \( x \) and such that

\[
\tag{3} \frac{m_1(E \cdot s)}{m_1(s)} > \frac{1}{2}.
\]

If \( E \) is a circle of radius \( R \), then clearly \( M(E) \) is a concentric circle of radius \( 3R \).

4. Lemma.

\[
\text{g.l.b.} \left| \frac{E}{M(E)} \right| = 0
\]

as \( E \) ranges over all plane sets.

If \( E \) is a circle, then \( |E|/|M(E)| = 1/9 \); if \( E \) is a triangle, then \( |E|/|M(E)| = 1/13 \).

Proof. It suffices to show that given an \( \varepsilon \) we can construct a set \( E \) such that

\[
\tag{4} \frac{|E|}{|M(E)|} < \varepsilon.
\]

Consider the open isosceles triangle \( S_0 = ABC \) of base \( BC = a \), angle \( BAC = \theta \), and altitude \( h \), with an axis \( \alpha - \alpha \) parallel to the \( y \)-axis. Clearly the points of the trapezoid \( T_0 = BCED \), where \( DE \) is the line \( y = -h \), belong to \( M(S_0) \). Draw the lines \( y = h - h' \) and \( y = h + h' \)

\footnote{This construction is similar to the one given by H. Busemann and W. Feller in [1].}

\footnote{The number \( m_1(s) \) will mean the linear measure of the set \( s \).}

\footnote{The above lemma is based on a construction used by O. Perron in giving a simple solution to the Besicovitch-Kakeya problem; it was utilized also by H. Busemann and W. Feller in their treatment of the density theorem.}
(\( h' \) to be determined soon); they intersect \( AB \) and \( AC \) and their extensions at the points \( K, L \) and \( L', K' \); the lines \( KK' \) and \( LL' \) intersect \( BC \) at the points \( M \) and \( N \). The open figure \( S_1 = AK'KBCLL'A \) has an area \( |S_1| = |S_0| \left( 1 + 2(h'/h)^2 \right) \); the line \( y = -(h+h') \) intersects the extensions of \( AB \) and \( AC \) at the points \( D_1 \) and \( E_1 \) forming the trapezoid \( T_1 = BCE_1D_1 \); clearly \( |T_1| > |T_0| (1 + h'/h) \) and \( T_1 \subset M(S_1) \).

With \( h' = h/2 \) we have

\[
(5) \quad |S_1| = |S_0| \left( 1 + 2(1/2)^2 \right), \quad |T_1| > |T_0| \{ 1 + 1/2 \}.
\]

We repeat the above construction with each of the triangles \( K'MC \) and \( L'BN \) with \( h' = h/3 \); we thus obtain the open figure \( S_2 \) consisting of \( S_1 \) and the four added triangles \( II \), and the trapezoid \( T_2 = BCE_2D_2 \). Again
Repeating the above process on each of the four new triangles, whose base is part of $BC$, with $h' = h/4$, we obtain the open figure $S_3$, consisting of $S_2$ and the eight added triangles, and the trapezoid $T_3 = BCE_3D_3$; as before

$$T_3 \subset M(S_3),$$

$$|S_3| = |S_0| \{1 + 2(1/2)^2 + 2(1/3)^2\},$$

$$|T_3| > |T_0| \{1 + 1/2 + 1/3\}.$$

At the $n$th such step with $h' = h/n$, we obtain the open figure $S_n$ and the trapezoid $T_n = BCE_nD_n$; they satisfy the relation

$$T_n \subset M(S_n),$$

$$|S_n| = |S_0| \{1 + 2(1/2)^2 + \cdots + 2(1/n)^2\},$$

$$|T_n| > |T_0| \{1 + 1/2 + \cdots + 1/n\}.$$

Since $1 + 1/2 + \cdots + 1/n + \cdots$ diverges, $1 + (1/2)^2 + \cdots + (1/n)^2 + \cdots$ converges, and $|T_0| \neq 0$, it follows that for a certain $n = N(\varepsilon)$ we shall have

$$\frac{|S_n|}{|S_n + T_n|} < \varepsilon,$$

and since $S_n + T_n \subset M(S_n)$, (4) has been established and the lemma is proved.

5. The set $S_n + T_n$ will be denoted by $B(\varepsilon; \theta)$, where $\varepsilon$ and $\theta$ are as in the previous section, and the set $S_n$ by $S(\varepsilon; \theta)$. Thus

$$\frac{|S(\varepsilon; \theta)|}{|B(\varepsilon; \theta)|} < \varepsilon, \quad B(\varepsilon; \theta) \subset M(S(\varepsilon; \theta)).$$

**Remark 1.** Obviously $\varepsilon$ and $\theta$ can be taken arbitrarily small in the above construction.

6. **Construction of $E$.** In this section $\theta$ will be kept constant and will be omitted in the expressions $S(\varepsilon; \theta)$ and $B(\varepsilon; \theta)$.

**First step.** By $[B(\varepsilon)]$ we shall mean the class of all sets similar to $B(\varepsilon)$ and similarly placed, with diameter smaller than one. For every $x \in \mathbb{Q}$, there exists a sequence of sets belonging to $[B(\varepsilon)]$, containing the point $x$ and with diameter tending to zero, hence (Vitali covering
theorem) there exists a sequence \( \{B(\varepsilon)\} \) of disjoint sets belonging to 
\( \{B(\varepsilon)\} \) and covering \( Q \) p.p. The sets \( S(\varepsilon) \) which are parts of \( B(\varepsilon) \) of
the sequence \( \{B(\varepsilon)\} \) form another sequence \( \{S(\varepsilon)\} \) whose sum we
denote by \( E_0 \); from (8) and the disjointness of the sets \( B(\varepsilon) \) of
\( \{B(\varepsilon)\} \) we conclude that
\[
|E_0| < \varepsilon |Q| = \varepsilon.
\]

Second step. By \( \{B(\varepsilon/2)\} \) we shall mean the class of all sets similar
to \( B(\varepsilon/2) \) and similarly placed with diameter less than \( 1/2 \). As before
there exists a sequence \( \{B(\varepsilon/2)\} \) of disjoint sets belonging to
\( \{B(\varepsilon/2)\} \) and covering \( Q \) p.p.; the sets \( S(\varepsilon/2) \) which are parts of
\( B(\varepsilon/2) \) of the sequence \( \{B(\varepsilon/2)\} \) form another sequence \( \{S(\varepsilon/2)\} \)
whose sum we denote by \( E_1 \); as before,
\[
|E_1| < \frac{\varepsilon}{2} |Q| = \varepsilon/2^1.
\]

\((k+1)\)th step. By \( \{B(\varepsilon/2^k)\} \) we shall mean the class of all sets similar
to \( B(\varepsilon/2^k) \) and similarly placed, with diameter smaller than \( 2/2^k \). Again there exists a sequence \( \{B(\varepsilon/2^k)\} \) of disjoint sets belonging to
\( \{B(\varepsilon/2^k)\} \) and covering \( Q \) p.p., and the sequence \( \{S(\varepsilon/2^k)\} \) of the
\( S(\varepsilon/2^k) \) sets whose sum we denote by \( E_k \); as before
\[
|E_k| < \varepsilon/2^k
\]
and so we continue.

Since we repeat the process an enumerable number of times, the
set of points of \( Q \) which are not covered at least once in this process
has a zero measure; hence it can be covered by an open set \( G \) of area
smaller than \( \varepsilon \); with \( E = G + \sum_{k=0}^{\infty} E_k \), we have
\[
|E| \leq |G| + \sum_{k=0}^{\infty} |E_k| < \varepsilon + \sum_{k=0}^{\infty} \frac{\varepsilon}{2^k} = 3\varepsilon.
\]

7. We shall now show that \( E \) satisfies the requirements of §2. Suppose \( x \in Q \); if \( x \) belongs to \( E \), \( E \) being open, we can cover \( x \) with a
circle \( C_x \subseteq E \). If we take any sequence \( \{I_n\} \) of intervals included in
\( C_x \) and with diameter tending to zero, we have \( |E \cdot I_n| / |I_n| = 1 > 1/3 \);
hence (2A) is true. Suppose \( x \) is not a point of \( E \); it suffices to show
that given \( \delta > 0 \) there exists an interval \( I_k \) of diameter smaller than \( \delta \)
and such that
\[
x \in I_k, \quad \frac{|E \cdot I_k|}{|I_k|} > \frac{1}{3}.
\]
Take \( k \) such that \( 1/2^k < \delta \); since \( x \in E \) it follows that \( x \in G \), hence it is covered by one set \( B(\epsilon/2^k) \) belonging to the sequence \( \{ B(\epsilon/2^k) \} \) of the \((k+1)\)th step. From the way these sets have been constructed, it follows that there exists a line segment \( s \) such that

\[
\forall \epsilon \in S(\epsilon/2^k), \quad s \subset B \left( \frac{\epsilon}{2^k} \right), \quad \frac{m_1(s \cdot S(\epsilon/2^k))}{m_1(s)} > \frac{1}{2}
\]

where \( S(\epsilon/2^k) \) is the set contained in \( B(\epsilon/2^k) \). Since the maximum diameter of the sets \( \{ B(\epsilon/2^k) \} \) is smaller than \( 1/2^k \), and \( s \subset B(\epsilon/2^k) \), we conclude that \( ds < 1/2^k \). But \( S(\epsilon/2^k) \subset E_k \subset E \), hence also

\[
\frac{m_1(s \cdot E)}{m_1(s)} > \frac{1}{2}
\]

and since \( E \) is open, we can find an interval \( I_k \) containing \( s \) and satisfying (15); and so our contention is proved.

**Remark 2.** Since the orientation of the \( I_k \)'s can be taken within the angle \( \theta \), and since \( \theta \) can be chosen arbitrarily small, it follows that the rectangles of the sequence \( \{ I_k \} \) in (2A) can be chosen with an orientation within an arbitrarily small angle.

8. With \( F = Q - E \) we have from (2A) with \( x \in F \subset Q \)

\[
\liminf_{I_n} \frac{|F \cdot I_n|}{|I_n|} < \frac{2}{3}.
\]

Hence (1) is not true because now we have a sequence \( \{ I_n \} \) of rectangles and not of intervals.

9. **Part B.** We shall construct a set \( E \) included in the unit square \( Q \) such that:

\[
|E| < 5\epsilon \quad \text{where} \quad \epsilon \text{ is an arbitrary number}; \quad \text{given} \quad x \in Q \text{ there exists a sequence} \quad \{ I_n \} \text{ of rectangles of "fixed orientation," containing the point} \quad x, \quad \text{with diameter tending to zero, and such that}
\]

\[
\limsup_{I_n} \frac{|E \cdot I_n|}{|I_n|} > \frac{1}{3}.
\]

10. **Notations.** Suppose angle \( BAC = \theta_1 \), angle \( EDF = \theta_2 \), and \( \gamma \) a given direction; if from \( A \) we draw a line parallel to \( \gamma \) and it falls inside \( BAC \), this will be indicated by writing \( \gamma \subset \theta_1 \). If from \( A \) we draw two lines parallel to \( DE \) and \( DF \) respectively and both fall inside \( BAC \), this will be indicated by writing \( \theta_2 \subset \theta_1 \). By \( s_\gamma \) we shall mean a line segment parallel to the direction \( \gamma \).

* \( ds \) will mean the diameter of the set \( s \).
11. **Outline of the method.** In the construction of Part A the direction of the line segments satisfying (14), and hence of the corresponding rectangles $I_k$, cannot be the same for every $k$ for the following reason: for a given $x \in \Omega$ the segments $s$ satisfying (14) for $k = 0$ may have any direction $\gamma \subset \delta^0$; for $k = 1$ may have any direction $\gamma \subset \delta^1$, for any $k$ a direction $\gamma \subset \delta^k$; the angle $\delta^k$ depends on the location of $x$ inside $B(\epsilon/2^k; \theta)$. It is possible that there is no direction $\gamma \subset \delta^k$ for every $k$. If we modify our second covering in such a manner that the covering sets $B(\epsilon/2; \theta')$ have an angle $\theta' \subset \delta^0$, then all the directions which satisfy (14) for $k = 1$ must be included in $\delta^1$, hence in $\delta^0$, they therefore must satisfy (14) also for $k = 0$; these possible directions form a nonzero angle $\theta^k$ which is used for the following step. Thus for each point $x$ we have a sequence of closed angles $\theta^0 \supset \theta^1 \supset \theta^2 \cdots$, hence there exists a $\gamma \subset \theta^k$ and a segment $s$, satisfying (14) for every $k$; and that is our objective.

12. We now come to our construction.

**First step.** We begin as in §6; we thus have the sequences $\{B(\epsilon; \theta)\}$, $\{S(\epsilon; \theta)\}$ and the set $E_1$ with

$$\left| E_1 \right| < \epsilon. \quad (9)$$

For almost every $x \in \Omega$ there exists an $s_{\gamma_0}$ of direction $\gamma_0$ and such that

$$x \in s, \quad ds < 1, \quad \frac{m_1(s \cdot E_1)}{m_1(s)} > \frac{1}{2}. \quad (16)$$

Since $E_1$ is open, it follows that all the possible directions $\gamma$ obtained by continuous rotation of $\gamma_0$, and for which the above is true, form a nonzero angle $w(x)$ depending on $x$. Consider the points $x$ of $Q$ such that $w(x) > \alpha$ where $\alpha$ is a given angle; they form a set $P_{\alpha}$; since to almost every $x$ of $Q$ there corresponds a nonzero angle $w(x)$, $\lim P_{\alpha}$ as $\alpha$ tends to zero must include almost all points of $Q$. Therefore given an $\epsilon$ there exists an $\alpha_1$ such that

$$\left| Q - P_{\alpha_1} \right| < \epsilon. \quad (17)$$

The points of $Q - P_{\alpha_1}$ can be included in an open set $G_0$ of measure less than $\epsilon$; with $R_1 = E_1 + G_1$ we have

$$\left| R_1 \right| < \epsilon + \epsilon = 2\epsilon. \quad (18)$$

Consider all points $x$ of $Q$ such that for every $\gamma \subset \theta$ there exists an $s$ such that (16) is true; they form a set which we denote by $1A_0^1$; clearly all points of $E_1$ belong to $1A_0^1$ since their corresponding $w$ equals $2\pi$. 

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In the following, angles will be considered with their orientation; thus when we speak of the set \( B(\varepsilon; \phi) \) we shall mean a set as in §5 but with its axis \( \alpha - \alpha \) not parallel to the \( y \)-axis but to the bisector of the angle \( \phi \).

If we bisect \( \theta \), we obtain the angles \( \theta_1^1 \) and \( \theta_2^1 \). We denote by \( 1A_1^1 \) (\( t = 1, 2 \)) the set of points of \( Q - 1A_0^1 \) such that for every \( \gamma \subset \theta_1^1 \) (\( t = 1, 2 \)) there exists an \( s_\gamma \) satisfying (16). We continue as follows:

We divide \( \theta \) into \( 2^n \) (\( n = 2, 3, \ldots \)) equal angles \( \theta_n^t \) (\( t = 1, 2, \ldots, 2^n \)) and denote by \( 1A_n^t \) (\( t = 1, \ldots, 2^n \)) the set of points of \( Q - \left\{ 1A_1^1 + 1A_1^2 + \cdots + 1A_{n-1}^1 + \cdots + 1A_{n-1}^{2^n} \right\} \) with the property that for every \( \gamma \subset \theta_n^t \) there exists an \( s_\gamma \) satisfying (16). After \( n_1 \) such steps, with \( \theta/2^{n_1+1} > \alpha_1 \), all points of \( Q - G_0 \) will be included in the sets \( 1A_h^t \) (\( h = 0, 1, \ldots, n_1; t = 1, 2, \ldots, 2^n \)) (see (17)).

To avoid three indices let us enumerate the sets \( 1A_h^t \) and their corresponding angles \( \theta_n^t \) writing \( 1A(i) \) and \( 10(i) \) with \( i = 1, 2, \ldots, N_1 \).

Kth step. Assuming that we have defined \( (k-1)A(i) \), \( (k-1)\theta(i) \), \( R_{k-1} \), and \( E_{k-1} \), we define \( kA(i) \), \( k\theta(i) \), \( R_k \), and \( E_k \) as follows: we cover p.p. the open kernel of \( (k-1)A(i) \) (which has the same measure as \( (k-1)A(i) \) as we can easily see) with a sequence \( \{ B(\varepsilon/2^{k-1}; (k-1)\theta(i)) \} \) of disjoint sets similar to \( B(\varepsilon/2^{k-1}; (k-1)\theta(i)) \) and similarly placed. The sets \( S \) which are parts of the \( B \)’s form another sequence \( \{ S(\varepsilon/2^{k-1}; (k-1)\theta(i)) \} \) whose sum we denote by \( E_k \); clearly

\[
|E_k| < \frac{\varepsilon}{2^{k-1}}.
\]

The subdivision of \( Q \) in the first step into the sets \( 1A_h^t \) we now perform on \( (k-1)A(i) \) for every \( i \), the only change being that instead of (16) we now must satisfy

\[
(20) \quad x \in s, \quad ds < \frac{1}{2^{k-1}}, \quad \frac{m_1(s \cdot E_{k-1})}{m_1(s)} > \frac{1}{2}.
\]

We thus obtain the sets \( kA(i)_h \) and the angles \( k\theta(i)_h \); we continue this process until we cover all points of the \( (k-1)A(i) \) except for a set of measure less than \( \varepsilon/2^{k-1} \) (see (17)) which we include in an open set \( G_k \) of measure less than \( \varepsilon/2^{k-1} \); and with \( R_k = E_k + G_k \) we have

\[
|R_k| < \frac{\varepsilon}{2^{k-1}} + \frac{\varepsilon}{2^{k-1}} = \frac{\varepsilon}{2^{k-2}}.
\]
The sets $kA(i)_h$ and $k\theta(i)_h$ we again enumerate for all $i$, $h$, and $t$, and we thus obtain $kA(i)$ and $k\theta(i)$.

We continue similarly. In this process the part of $Q$ which is not covered at least once by the sets $\{B\}$ has a zero measure; we cover it with a set $G$ of area smaller than $\varepsilon$; with $E = G + \sum_{k=1}^n R_k$, we have $|E| < \varepsilon + \sum_{k=1}^n \varepsilon/2^{k-2} = 5\varepsilon$.

14. We shall now prove that $E$ satisfies (2B). Suppose $x \in Q$ if $x \in G_k + G$; then we have no problem since $x$ can be covered by a circle included in $E$.

Suppose then that $x \in G_k + G$; then $x \in B(\varepsilon/2^{k-1}; (k-1)\theta(i))$ for a certain $i$ depending on $x$ (and $k$), and every $k$. From the way the different coverings have been performed, it follows that

$$1\theta(i) \supset 2\theta(i) \supset \cdots \supset (k-1)\theta(i) \supset \cdots$$

and since these angles are closed there must exist a direction $\gamma_s$ included in all of them and such that (20) is true for every $k$ if the segment $s$ is parallel to $\gamma_s$; from that and the openness of $E$, (2B) follows easily.

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