ON ROUCHÉ'S THEOREM AND THE INTEGRAL-SQUARE MEASURE OF APPROXIMATION

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1. Introduction. A well known theorem due to Hurwitz asserts that if the sequence of functions $f_n(z)$ analytic in a closed region $R$ converges uniformly in $R$ to the function $f(z)$ which does not vanish on the boundary $B$ of $R$, then for $n$ sufficiently large the functions $f(z)$ and $f_n(z)$ have the same number of zeros in $R$. Hurwitz's theorem may be applied either to $R$ or to mutually disjoint neighborhoods $N(z_k)$ in $R$ of the distinct zeros $z_k$ of $f(z)$ in $R$; for $n$ sufficiently large, each $N(z_k)$ contains the same number of zeros of $f_n(z)$ as of $f(z)$, and no zeros of $f_n(z)$ lie in $R$ exterior to the $N(z_k)$.

Hurwitz's theorem is ordinarily proved from the theorem of Rouché: If $f(z)$ and $F(z)$ are analytic in a region $R$ whose boundary is $B$, and if we have on $B$ the relations $f(z) \neq 0$ and

$$\left| \frac{f(z) - F(z)}{f(z)} \right| < 1,$$

then $f(z)$ and $F(z)$ have the same number of zeros in $R$. A less precise but qualitatively identical theorem can be proved by Hurwitz's theorem: If a function $f(z)$ analytic in $R$ is different from zero on $B$, there exists a number $\delta$ ($>0$) depending on $f(z)$ and $R$ such that the inequality $|f(z) - F(z)| < \delta$ on $B$ for a function $F(z)$ analytic in $R$ implies that $f(z)$ and $F(z)$ have the same number of zeros in $R$. If this statement is false, there exist functions $F_n(z)$ analytic in $R$ with

$$|f(z) - F_n(z)|<1/n \text{ in } R,$$

where $F_n(z)$ and $f(z)$ do not have the same number of zeros in $R$; the sequence $F_n(z)$ converges uniformly to $f(z)$ in $R$, and this contradicts Hurwitz's theorem. Of course it follows from Rouché's theorem that we may choose $\delta = \min |f(z)|$ on $B$.

Thus Hurwitz's theorem and Rouché's theorem are intimately connected with each other and with the measure of approximation

$$\max |f(z) - F(z)|, \quad z \text{ in } R,$$

as metric; this formulation suggests the corresponding study of other measures of approximation, such as the metric

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where $B$ is assumed rectifiable and $R$ is bounded. For functions $f(z)$ and $f_n(z)$ analytic in $R$ the relation
\[ \int_B |f(z) - f_n(z)|^2 \, dz \to 0 \]
does not imply the uniform convergence of the sequence $f_n(z)$ to $f(z)$ in $R$, but (by use of Cauchy's integral formula) does imply such uniform convergence in every closed subregion of $R$ interior to $R$. Thus, by a repetition of the reasoning already set forth, it follows that if $f(z)$ and $R$ are given, if we choose a closed subregion $R_1$ of $R$ interior to $R$ on the boundary of which $f(z)$ is different from zero, and if we choose disjoint neighborhoods $N(z_k)$ in $R_1$ of the distinct zeros $z_k$ of $f(z)$ in $R_1$, then there exists a number $\delta_1 (>0)$ such that the inequality
\[ \int_B |f(z) - F(z)|^2 \, dz < \delta_1 \]
implies that $F(z)$ has precisely the same number of zeros in $R_1$ and in each $N(z_k)$ as does $f(z)$, and $F(z)$ has no zeros in $R_1$ exterior to the $N(z_k)$; here $\delta_1$ depends on $f(z)$, $R$, $R_1$, and the $N(z_k)$, but not on $F(z)$. Indeed, if we use Cauchy's integral formula
\[ [f(z) - F(z)]^2 = \frac{1}{2\pi i} \int_B \frac{[f(t) - F(t)]^2 \, dt}{t - z}, \quad z \in R, \]
it follows from Rouché's theorem that we may choose
\[ \delta_1 = 2\pi d \cdot \min |f(z)|^2 \text{ on boundary of } R_1 + \sum N(z_k), \]
where $d$ is the distance from $B$ to $R_1$.

The object of the present note is to study the analogue of Rouché's theorem, and in particular to determine the best number $\delta_1$ in the simplest nontrivial cases, namely that where $R$ is the unit circle and $f(z)$ is a power of $z$.

Condition (3) does not imply the uniform convergence of $f_n(z)$ to $f(z)$ in $R$, so it is not to be expected that $f_n(z)$ and $f(z)$ necessarily have the same number of zeros in $R$.

**Theorem 1.** Let $\mu$ be a given non-negative integer, and let positive numbers $\epsilon$ and $N$ be given with $N$ integral. Then there exists a function $\psi(z)$ analytic for $|z| \leq 1$, with precisely $\mu + N$ zeros in $|z| < 1$, such that
We set
\[
\psi(z) = z^n \left( \frac{z + \alpha}{1 + az} \right)^N,
\]
where \(\alpha\) is to be further restricted later. On \(C\) we have \(|\psi(z)| = 1\), so the first member of (4) can be written
\[
\int_C \left| \frac{z + \alpha}{1 + az} \right|^2 \, dz = 4\pi [1 - \alpha^N],
\]
and this last expression is less than \(\epsilon\) if \(\alpha\) is chosen sufficiently near unity.

We denote by \(H_2\) the class of functions \(\sum a_n z^n\) analytic interior to \(C\), with \(\sum |a_n|^2\) convergent; it is then well known that boundary values for normal approach exist almost everywhere on \(C\), and are integrable and square-integrable (Lebesgue) on \(C\), and that Cauchy’s integral formula is valid. We introduce the notation
\[
[f(z), F(z)] = \frac{1}{2\pi} \int_C |f(z) - F(z)|^2 \, dz = \sum_0^\infty |c_n|^2,
\]
where the function \(f(z) - F(z) = \sum_0^n c_n z^n\) is assumed of class \(H_2\). To Theorem 1 we add the

**Corollary.** Let \(\mu\) be a given non-negative integer, and let \(\epsilon\) (>0) be given. Then there exists a function \(\psi(z)\) of class \(H_2\), with infinitely many zeros in \(|z| < 1\), such that we have
\[
[f(z), \psi] < \epsilon.
\]

We set
\[
B_n(z) = \prod_{1}^{n} \frac{\bar{\beta}_k}{|\beta_k|} \frac{z - \beta_k}{\bar{\beta}_k z - 1}, \quad |z| < 1,
\]
with \(|\beta_k| < 1\), where the \(\beta_k\) are to be further restricted later. We then have\(^1\)

\([z^n B_n(z), z^n] = [B_n(z), 1] = 4\pi [1 - |\beta_1\beta_2 \cdots \beta_n|]\).

If the \(\beta_k\) are chosen infinite in number so that the corresponding Blaschke product

\[B(z) = \lim_{n \to \infty} B_n(z)\]

converges \((|z| < 1)\), the sequence \(B_n(z)\) converges (loc. cit.) in the mean on \(C\) to the boundary values of \(B(z)\), and we have with \(\psi(z) = z^n B(z)\)

\([\psi(z), z^n] = 4\pi [1 - |\beta_1\beta_2 \cdots |];\)

the second member is less than \(\varepsilon\) if the \(\beta_k\) are suitably chosen.

2. Main theorems. We turn to an analogue of Rouché’s theorem for \(d(z) \equiv 1\).

**Theorem 2.** Let \(f(z)\) be of class \(H_2\), with

\[|f(z), 1| < (1 - r^2), \quad 0 < r < 1;\]

then \(f(z)\) has no zeros in the closed region \(|z| \leq r\).

We set \(f(z) = \sum_0^\infty a_n z^n\). By Cauchy’s algebraic inequality we have for \(|z| = r\)

\[|f(z) - 1| \leq |a_0 - 1| + \sum_1^\infty |a_n z^n|\]

\[\leq \left[ |a_0 - 1|^2 + \sum_1^\infty |a_n|^2 \right]^{1/2} \cdot \left[ \sum_0^\infty |z^n|^2 \right]^{1/2}.\]

Of course we have, by the orthogonality properties of the powers of \(z\),

\[|f(z), 1| = \left[ |a_0 - 1|^2 + \sum_1^\infty |a_n|^2 \right],\]

whence on \(|z| = r\)

\[|f(z) - 1| < 1;\]

it follows that \(f(z)\) has no zeros in \(|z| \leq r\).

The conclusion of Theorem 2 is not valid if we replace the second member of (5) by any larger number, for if we set \(f(z) = r(r - z)/(1 - rz)\), which has a zero on the circle \(|z| = r\), the first member of (5) is \((1 - r^2)\).
Theorem 2 is essentially a limiting case of

**Theorem 3.** Let $\mu$ be a positive integer, and suppose for some $r$ ($0 < r < 1$) we have

$$[f(z), z^\mu] < r^{2\mu}(1 - r^2),$$

where $f(z)$ is of class $H_2$; then $f(z)$ has precisely $\mu$ zeros in the region $|z| < r$ and no zero on the circle $|z| = r$.

More explicitly, suppose we have

$$0 < \varepsilon < \varepsilon_\mu = \frac{\mu^\mu}{(1 + \mu)^{1+\mu}},$$

and denote by $r_1$ and $r_2$ ($0 < r_1 < r_2 < 1$) zeros of the equation

$$(7) \quad r^{2\mu}(1 - r^2) = \varepsilon.$$

If $f(z)$ is of class $H_2$, and if we have

$$(8) \quad |f(z), z^\mu| < \varepsilon,$$

then $f(z)$ has precisely $\mu$ zeros in the region $|z| < r_1$ and no zeros in the closed annulus $r_1 \leq |z| \leq r_2$.

We prove the latter part of Theorem 3, which includes the former part. It follows from Descartes's rule of signs that equation (7) has no more than two positive roots $r_1$ and $r_2$; the first member of (7) vanishes for $r = 0$ and $r = 1$, is positive in the interval $0 < r < 1$, and has there the maximum value $\varepsilon_\mu$. The first member of (7) is greater than $\varepsilon$ in the interval $r_1 < r < r_2$.

If we set $f(z) \equiv \sum_0^\infty a_n z^n$, we have for $|z| = r < 1$ by Cauchy's inequality

$$|f(z) - z^\mu| \leq \left[ |a_0|^2 + |a_1|^2 + \cdots + |a_{\mu-1}|^2 + |a_\mu - 1|^2 + |a_{\mu+1}|^2 + \cdots \right]^{1/2} \left[ \sum_0^\infty |z^n|^2 \right]^{1/2}.$$

We also have

$$[f(z), z^\mu] = \left[ |a_0|^2 + |a_1|^2 + \cdots + |a_\mu - 1|^2 + \cdots \right],$$

whence on $|z| = r$ ($< 1$)

$$\frac{|f(z) - z^\mu|}{z^\mu} < \frac{\varepsilon^{1/2}}{r^{2\mu}(1 - r^2)^{1/2}},$$

and this last member is not greater than unity for $r$ in the closed interval $r_1 \leq r \leq r_2$. The conclusion of Theorem 3 follows from Rouché's theorem.
The latter part of Theorem 3 is not valid if we replace the second member of (8) by any larger number, for if we set \( f(z) = z^n - r^n(1-r^2)/(1-rz) \), the function \( f(z) \) has a zero \( z=r \) on the circle \( |z|=r \); for this function the first member of (8) is \( r^{2n}(1-r^2) \).

At least so far as concerns approximation on \( C \) to the functions \( z^n, n \geq 0 \), Theorems 1, 2, and 3 give a complete solution to the problem proposed, namely the investigation of (2) as a measure of approximation, with reference to the number and location of the zeros of the approximating functions interior to \( C \). These results have been established by the use of Rouché's theorem itself and standard methods; no new principle to replace Rouché's theorem is needed here.

The application of Theorem 2 in the study of a specific function \( F(z) \) of class \( H_2 \) is not unique, for we may set \( f(z) = AF(z) \), where \( A \) is an arbitrary constant. It is natural to choose \( A \) so that \( |AF(z), 1| \) is as small as possible; thus if we have \( F(z) = \sum a_n z^n \), we should minimize

\[
|AF(z), 1| = |Aa_0 - 1|^2 + |Aa_1|^2 + |Aa_2|^2 + \cdots.
\]

It is clear that for given \( |A| \) we should choose \( \arg A \) so that \( Aa_0 \) is positive (we ignore the trivial case \( a_0 = 0 \)), so we have \( |Aa_0 - 1| = |A| |a_0| - 1 \). The minimum for all \( |A| \) of the function

\[
|A| |a_0| - 1 + |A|^2 |a_1|^2 + |A|^2 |a_2|^2 + \cdots
\]

occurs for \( |A| = |a_0| / [|a_0|^2 + |a_1|^2 + \cdots] \) and equals

\[
1 - |a_0|^2 / [|a_0|^2 + |a_1|^2 + \cdots].
\]

Inequality (5) then takes the form

\[
r^2 < |a_0|^2 / [|a_0|^2 + |a_1|^2 + \cdots].
\]

It follows that if the function \( F(z) = \sum a_n z^n \) with \( a_0 \neq 0 \) is of class \( H_2 \), then \( F(z) \) has no zero in the region

\[
|z| < |a_0| / [|a_0|^2 + |a_1|^2 + \cdots]^{1/2}.
\]

This result is due to Petrovitch, and was later studied also by Landau.

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1 - \left| a_\mu \right|^2 / \left( \left| a_0 \right|^2 + \left| a_1 \right|^2 + \cdots \right) < \epsilon < \epsilon_n,

then \( f(z) \) has precisely \( \mu \) zeros in the region \( |z| < r_1 \) and no zeros in the closed annulus \( r_1 \leq |z| \leq r_2 \).

Theorem 3 is closely analogous to the well known theorem of Pellet, that the condition

\[ |a_k| r^k > |a_0| + |a_1| r + \cdots + |a_{k-1}| r^{k-1} \]

\[ + |a_{k+1}| r^{k+1} + \cdots + |a_n| r^n \]

implies that the polynomial \( \sum_n a_n z^n \) has precisely \( k \) zeros in the region \( |z| < r \); Pellet's theorem applies also to a power series converging uniformly for \( |z| = r \).

3. Extremal functions. For the sake of completeness we determine the extremal functions:

**Theorem 4.** If the hypothesis of Theorem 2 is modified by replacing the sign \( < \) in (5) by the sign \( \leq \), then either \( f(z) \) has no zeros in the closed region \( |z| \leq r \) or \( f(z) \) is of the form \( 1 - (1 - r^2) / (1 - \gamma r z) \) with \( |\gamma| = 1 \).

If the hypothesis of Theorem 3 is modified by replacing the sign \( < \) in (8) by the sign \( \leq \), then either \( f(z) \) has precisely \( \mu \) zeros in the region \( |z| < r_1 \) and no zeros in the closed annulus \( r_1 \leq |z| \leq r_2 \), or \( f(z) \) is of the form \( z^n - r_j^n (1 - r_j^2) / \gamma_j^n (1 - \gamma_j r z) \) with \( |\gamma_j| = 1, j = 1 \) or 2.

To establish the first part of Theorem 4, we merely notice that the original proof of Theorem 2 (in particular the use of Cauchy's inequality) remains valid under the modified hypothesis unless the two sets of numbers

\[ a_0 - 1, a_1, a_2, \ldots, 1, z, z^2, \ldots \]

are each proportional to the conjugate of the other, for some \( z = z_0, |z_0| = r \):

\[ a_0 - 1 = \lambda, \quad a_1 = \lambda z_0, \quad a_2 = \lambda z_0^2, \ldots . \]

Here we have (in any case under Theorem 4 for which the conclusion of Theorem 2 is not satisfied)

\[ \left| a_0 - 1 \right|^2 + \sum_1^\infty \left| a_n \right|^2 = 1 - r^2, \]

\[ \left| \lambda \right|^2 \sum_0^\infty \left| z_0 \right|^{2n} = 1 - r^2, \]

\[ \left| \lambda \right| = 1 - r^2. \]
Moreover we have \(|z| < 1\)

\[ f(z) = (1 + \lambda) + \lambda \bar{z}_0 z + \lambda \bar{z}_0^2 z^2 + \cdots = 1 + \lambda/(1 - \bar{z}_0 z), \]

whose only zero is \(z = (1 + \lambda)/\bar{z}_0\); the modulus of this zero is \(|1 + \lambda|/r\), which is not less than \((1 - |\lambda|)/r = r\) and is equal to \(r\) when and only when we have \(\lambda = -(1 - r^2)\).

The latter part of Theorem 4 is similarly proved. In any case under Theorem 4 not included in the original hypothesis, the original proof is valid unless we have for some \(z = z_0, \ |z_0| = r_0, j = 1 \) or \(2,\)

\[ a_0 = \lambda, \ a_1 = \lambda \bar{z}_0, \ \ldots, \ a_{\mu - 1} = \lambda \bar{z}_0^{\mu - 1}, \ a_\mu - 1 = \lambda \bar{z}_0^\mu, \ a_{\mu + 1} = \lambda \bar{z}_0^{\mu + 1}, \ \ldots. \]

Thus we have in any exceptional case

\[ |a_0|^2 + |a_1|^2 + \cdots + |a_\mu|^2 + |a_{\mu + 1}|^2 + \cdots = r_j^2(1 - r_j^2), \]

\[ |\lambda|^2 \sum_{j=0}^{\infty} |z_0|^{2n} = r_j^2(1 - r_j^2), \]

\[ |\lambda| = r_j^\mu(1 - r_j^2). \]

Moreover we have \(|z| < 1\)

\[ f(z) = \lambda + \lambda \bar{z}_0 z + \cdots + \lambda \bar{z}_0^{\mu - 1} z^{\mu - 1} + (1 + \lambda \bar{z}_0) z^{\mu} \]

\[ + \lambda \bar{z}_0^{\mu + 1} z^{\mu + 1} + \cdots \]

\[ = z^n + \lambda/(1 - \bar{z}_0 z). \]

The zeros of \(f(z)\) are the zeros of \(\bar{z}_0 z^{n+1} - z^n - \lambda\), and in such a zero we have \(|\lambda| = r_j^\mu(1 - r_j^2) = |\bar{z}_0 z^n - z^n| = |z^n| \cdot |1 - \bar{z}_0 z|\). Thus \(z\) is not a zero of \(f(z)\) on \(|z| = r_j\) unless we have \(z = z_0, \lambda = -z_0^\mu(1 - r_j^2)\); Theorem 4 is established.

We mention a further limiting case under Theorem 3: If \(f(z)\) is of class \(H_2\) and we have \([f(z), z^n] = e_n\), then either \(f(z)\) is of the form \(z^n - r^n(1 - r^2)/\gamma^n(1 - \gamma r^2)\) with \(r = r_0 = [\mu/(1 + \mu)]^{1/2}, \gamma = 1\), or \(f(z)\) has precisely \(\mu\) zeros in the region \(|z| < r_0\) and no zeros on the circle \(|z| = r_0\).

4. Polynomials. The methods already used apply also in the study of zeros of polynomials of given degree \(\nu\), namely functions of the form \(p(z) = \sum_0^\nu a_n z^n\). Here the circle \(|z| = 1\) is of no especial significance, but we continue to use the measure of approximation (\(\nu \geq \mu \geq 0\))

\[ [p(z), z^n] = |a_0|^2 + |a_1|^2 + \cdots + |a_{\nu - 1}|^2 + |a_\nu - 1|^2 + |a_{\nu + 1}|^2 + \cdots + |a_\nu|^2. \]
The analogue of Theorem 2 is

**Theorem 5.** Suppose \( p(z) = \sum_0^\infty a_n z^n \) with

\[
[p(z), 1] < 1/(1 + r^2 + r^4 + \cdots + r^{2n});
\]

then \( p(z) \) has no zeros in the closed region \( |z| \leq r \).

Theorem 5 is established by the same method as is Theorem 2; the proof is omitted. The analogue of Theorem 3 is

**Theorem 6.** Suppose \( p(z) = \sum_0^\infty a_n z^n \), suppose \( \mu(\nu) \) is a positive integer, and suppose

\[
[p(z), z^n] = A_\mu < r^{\nu}/(1 + r^2 + r^4 + \cdots + r^{2n});
\]

then \( p(z) \) has precisely \( \mu \) zeros in the region \( |z| < r \) and no zeros on the circle \( |z| = r \).

Consequently, if the equation

\[
1 + r^2 + r^4 + \cdots + r^{2\nu} - r^{\nu}/A_\mu = 0, \quad A_\mu \neq 0,
\]

has two positive zeros \( r_1 \) and \( r_2 \ (> r_1) \), then \( p(z) \) has no zeros in the annulus \( r_1 < |z| < r_2 \), and has precisely \( \mu \) zeros in the closed region \( |z| \leq r_1 \).

It follows by Descartes's rule of signs that (10) has no more than two positive zeros. Moreover the first member of (10) is positive for \( r = 0 \) and \( r \to +\infty \), so if (10) has two positive zeros as indicated, inequality (9) is satisfied in the interval \( r_1 < r < r_2 \). Theorem 6 follows by the Cauchy inequality

\[
| a_0 + a_1 z + \cdots + (a_\mu - 1)z^n + \cdots + a_n z^n |^2 \leq A_\mu (1 + r^2 + \cdots + r^{2n}),
\]

whence we have on the circle \( |z| = r, r_1 < r < r_2 \),

\[
\left| \frac{p(z) - z^n}{z^n} \right| < 1,
\]

and by Rouche's theorem.

A further result for polynomials, which gives an upper bound for the moduli of the zeros, and which has no analogue for arbitrary functions of class \( H_2 \), is

**Theorem 7.** Suppose \( p(z) = \sum_0^\infty a_n z^n \) and suppose

\[
[p(z), z^n] = A_\nu < r^{\nu}/(1 + r^2 + \cdots + r^{2n});
\]

then all zeros of \( p(z) \) lie in the region \( |z| < r \).
The proof of Theorem 7 is similar to that of Theorem 6, and can also be given from Theorem 5 by the substitution $w = 1/z$; it is left to the reader.

We have already indicated that it may be more favorable to apply Theorems 2 and 3 to the function $AF(z)$ rather than to a given function $F(z)$. A similar remark applies to Theorems 5-7. We formulate:

If we have $p(z) = \sum_{n=0}^{\infty} a_n z^n$ with
\[
1 - \left| a_0 \right|^2 / \left[ |a_0|^2 + |a_1|^2 + \cdots + |a_r|^2 \right] < 1/(1 + r^2 + r^4 + \cdots + r^{2r}),
\]
then $p(z)$ has no zeros in the closed region $|z| \leq r$.

If we have $p(z) = \sum_{n=0}^{\infty} a_n z^n$, $0 < \mu < \nu$, with
\[
1 - \left| a_\mu \right|^2 / \left[ |a_0|^2 + |a_1|^2 + \cdots + |a_r|^2 \right] < \frac{r^{2\mu}}{(1 + r^2 + r^4 + \cdots + r^{2r})},
\]
then $p(z)$ has precisely $\mu$ zeros in the region $|z| < r$ and no zero on the circle $|z| = r$. Consequently if the equation
\[
1 + r^2 + r^4 + \cdots + r^{2\mu} - r^{2\mu} \left[ |a_0|^2 + |a_1|^2 + \cdots + |a_\mu|^2 \right] / \left[ |a_0|^2 + |a_1|^2 + \cdots + |a_\mu|^2 + |a_{\mu+1}|^2 + \cdots + |a_r|^2 \right] = 0
\]
has two positive zeros $r_1$ and $r_2 (> r_1)$, then $p(z)$ has no zeros in the annulus $r_1 < |z| < r_2$ and has precisely $\mu$ zeros in the closed region $|z| \leq r_1$.

If we have $p(z) = \sum_{n=0}^{\infty} a_n z^n$, with
\[
1 - \left| a_r \right|^2 / \left[ |a_0|^2 + |a_1|^2 + \cdots + |a_r|^2 \right] < r^{2r}/(1 + r^2 + r^4 + \cdots + r^{2r}),
\]
then all zeros of $p(z)$ lie in the region $|z| < r$.

Theorems 5-7 contain bounds which cannot be improved. For the sake of completeness we determine the extremal functions:

**Theorem 8.** If the hypothesis of Theorem 5 is modified by replacing the sign $<$ by the sign $\leq$, then either $p(z)$ has no zeros in the closed region $|z| < r$ or we have $p(z) = 1 - (1-r^2)(1-\gamma^{r+1} r^{2r+1})/(1-r^{2r+2})(1-\gamma r z)$, with $|\gamma| = 1$.

If the hypothesis of Theorem 6 is modified by replacing in (9) the sign $<$ by the sign $\leq$, then either $p(z)$ has precisely $\mu$ zeros in the closed region $|z| \leq r$ or we have
\[
p(z) = a^\mu - \gamma^\mu (1-r^2)(1-\gamma^{r+1} r^{2r+1})/(1-r^{2r+2})(1-\gamma r z)
\]
with $|\gamma| = 1$. Consequently if (10) has two positive zeros $r_1$ and $r_2$, then either $p(z)$ has no zeros in the closed annulus $r_1 \leq |z| \leq r_2$ and has pre-
cisely μ zeros in the closed region |z| ≤ r₁, or p(z) is of the form

\[ z^\mu - r_j(1 - r_j)(1 - γ^{r+1})^{r+1} / γ^\mu (1 - r_j)^{2γ^r}(1 - γr_\mu z), \quad j = 1 \text{ or } 2. \]

If the hypothesis of Theorem 7 is modified by replacing in (11) the sign < by the sign ≤, then either p(z) has all its zeros in the region |z| < r or we have p(z) = z′ − r′(1−r′)(1−γ′+1r′+1)/γ′(1−γrz) \cdot (1−r^{2γ+2}) \text{ with } |γ| = 1.

5. Related problems. Theorems 2–4 can properly be viewed as analogues of Rouché’s theorem, for approximation on C to the functions z^{μ}, insofar as such analogues exist; sufficient conditions are derived that f(z) and z^{μ} should have the same number of zeros in a suitable region interior to C. Still another problem suggests itself, however:

Problem I. To determine the smallest number η_{μ} (μ > 0) such that for a function f(z) of class H_2 the inequality [f(z), z^{μ}] < η_{μ} implies that f(z) has at least one zero interior to C.

Problem I is still open, but it is clear that from Theorem 3 we have η_{μ} ≥ μ^2/(1+μ)^{1+μ}. Moreover for the specific function

\[ f_0(z) = -(z^μ - 1)^2(z^μ + 4)/10, \]

which has no zeros interior to C, we have [f_0(z), z^{μ}] = 3/10, whence η_{μ} ≤ 3/10.

Modifications of Problem I suggest themselves:

Problem II. To determine the smallest number η^{(β)}_{μ} such that for a function f(z) of class H_2 the inequality [f(z), z^{μ}] < η^{(β)}_{μ}, μ ≥ β > 0, implies that f(z) has at least β zeros interior to C.

Problem III. To determine the smallest number η^{(β)}_{μ}(ν) such that for a polynomial p(z) of degree ν the inequality

\[ [p(z), z^{μ}] < η^{(β)}_{μ}(ν), \quad ν ≥ μ ≥ β > 0, \]

implies that p(z) has at least β zeros interior to C.

A further obvious problem is to replace the metric [f(z), F(z)] by the new metric

\[ \frac{1}{2π} \int_C |f(z) - F(z)|^p \, |dz|, \quad p > 0; \]

Cauchy’s integral formula then provides a bound on |f(z) − F(z)| on the circle |z| = r < 1, but this new metric has no simple relation to the coefficients in the Taylor developments of f(z) and F(z).

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