ON ROUCHÉ'S THEOREM AND THE INTEGRAL-SQUARE MEASURE OF APPROXIMATION

J. L. WALSH

1. **Introduction.** A well known theorem due to Hurwitz asserts that if the sequence of functions $f_n(z)$ analytic in a closed region R converges uniformly in R to the function $f(z)$ which does not vanish on the boundary B of R, then for n sufficiently large the functions $f(z)$ and $f_n(z)$ have the same number of zeros in R. Hurwitz's theorem may be applied either to R or to mutually disjoint neighborhoods $N(z_k)$ in R of the distinct zeros z_k of $f(z)$ in R; for n sufficiently large, each $N(z_k)$ contains the same number of zeros of $f_n(z)$ as of $f(z)$, and no zeros of $f_n(z)$ lie in R exterior to the $N(z_k)$.

Hurwitz's theorem is ordinarily proved from the theorem of Rouché: *If $f(z)$ and $F(z)$ are analytic in a region R whose boundary is B, and if we have on B the relations $f(z) \neq 0$ and*

$$\left| \frac{f(z) - F(z)}{f(z)} \right| < 1, \tag{1}$$

then $f(z)$ and $F(z)$ have the same number of zeros in R. A less precise but qualitatively identical theorem can be proved by Hurwitz's theorem: *If a function $f(z)$ analytic in R is different from zero on B, there exists a number δ (>0) depending on $f(z)$ and R such that the inequality $|f(z) - F(z)| < \delta$ on B for a function $F(z)$ analytic in R implies that $f(z)$ and $F(z)$ have the same number of zeros in R.* If this statement is false, there exist functions $F_n(z)$ analytic in R with

$$|f(z) - F_n(z)| < 1/n \quad \text{in} \quad R,$$

where $F_n(z)$ and $f(z)$ do not have the same number of zeros in R; the sequence $F_n(z)$ converges uniformly to $f(z)$ in R, and this contradicts Hurwitz's theorem. Of course it follows from Rouché's theorem that we may choose $\delta = \min |f(z)|$ on B.

Thus Hurwitz's theorem and Rouché's theorem are intimately connected with each other and with the measure of approximation

$$\max |f(z) - F(z)|, \qquad z \text{ in } R,$$

as metric; this formulation suggests the corresponding study of other measures of approximation, such as the metric

---

Presented to the International Congress of Mathematicians, September 2, 1950; received by the editors July 24, 1950 and, in revised form, September 25, 1950.

$$\int_B |f(z) - F(z)|^2 |dz|, \tag{2}$$

where B is assumed rectifiable and R is bounded. For functions $f(z)$ and $f_n(z)$ analytic in R the relation

$$\int_B |f(z) - f_n(z)|^2 |dz| \to 0 \tag{3}$$

does not imply the uniform convergence of the sequence $f_n(z)$ to $f(z)$ in R, but (by use of Cauchy's integral formula) does imply such uniform convergence in every closed subregion of R interior to R. Thus, by a repetition of the reasoning already set forth, it follows that if $f(z)$ and R are given, if we choose a closed subregion R_1 of R interior to R on the boundary of which $f(z)$ is different from zero, and if we choose disjoint neighborhoods $N(z_k)$ in R_1 of the distinct zeros z_k of $f(z)$ in R_1, then *there exists a number* δ_1 (>0) *such that the inequality*

$$\int_B |f(z) - F(z)|^2 |dz| < \delta_1$$

implies that $F(z)$ *has precisely the same number of zeros in* R_1 *and in each* $N(z_k)$ *as does* $f(z)$, *and* $F(z)$ *has no zeros in* R_1 *exterior to the* $N(z_k)$; here δ_1 depends on $f(z)$, R, R_1, and the $N(z_k)$, but not on $F(z)$. Indeed, if we use Cauchy's integral formula

$$[f(z) - F(z)]^2 = \frac{1}{2\pi i}\int_B \frac{[f(t) - F(t)]^2 dt}{t - z}, \qquad z \text{ in } R,$$

it follows from Rouché's theorem that we may choose

$$\delta_1 = 2\pi d \cdot [\min |f(z)|^2 \text{ on boundary of } R_1 + \sum N(z_k)],$$

where d is the distance from B to R_1.

The object of the present note is to study the analogue of Rouché's theorem, and in particular to determine the best number δ_1 in the simplest nontrivial cases, namely that where R is the unit circle and $f(z)$ is a power of z.

Condition (3) does not imply the uniform convergence of $f_n(z)$ to $f(z)$ in R, so it is not to be expected that $f_n(z)$ and $f(z)$ necessarily have the same number of zeros in R.

THEOREM 1. *Let* μ *be a given non-negative integer, and let positive numbers* ϵ *and* N *be given with* N *integral. Then there exists a function* $\psi(z)$ *analytic for* $|z| \leq 1$, *with precisely* $\mu+N$ *zeros in* $|z| < 1$, *such that*

$$\int_C |\psi(z) - z^\mu|^2 |dz| < \epsilon. \tag{4}$$

We set

$$\psi(z) \equiv z^\mu \left(\frac{z+\alpha}{1+\alpha z}\right)^N, \qquad 0 < \alpha < 1,$$

where α is to be further restricted later. On C we have $|\psi(z)| \equiv 1$, so the first member of (4) can be written

$$\int_C \left| \left(\frac{z+\alpha}{1+\alpha z}\right)^N - 1 \right|^2 |dz|$$
$$= \int_C \left[2 - \left(\frac{z+\alpha}{1+\alpha z}\right)^N - \left(\frac{\bar z+\bar\alpha}{1+\bar\alpha\bar z}\right)^N \right] \frac{dz}{iz} = 4\pi[1-\alpha^N],$$

and this last expression is less than ϵ if α is chosen sufficiently near unity.

We denote by H_2 the class of functions $\sum a_n z^n$ analytic interior to C, with $\sum |az_n|^2$ convergent; it is then well known that boundary values for normal approach exist almost everywhere on C, and are integrable and square-integrable (Lebesgue) on C, and that Cauchy's integral formula is valid. We introduce the notation

$$[f(z), F(z)] \equiv \frac{1}{2\pi} \int_C |f(z) - F(z)|^2 |dz| \equiv \sum_0^\infty |c_n|^2,$$

where the function $f(z) - F(z) \equiv \sum_0^\infty c_n z^n$ is assumed of class H_2. To Theorem 1 we add the

Corollary. *Let μ be a given non-negative integer, and let ϵ (>0) be given. Then there exists a function $\psi(z)$ of class H_2, with infinitely many zeros in $|z| < 1$, such that we have*

$$[\psi(z), z^\mu] < \epsilon.$$

We set

$$B_n(z) \equiv \prod_1^n \frac{\bar\beta_k}{|\beta_k|} \frac{z-\beta_k}{\bar\beta_k z - 1}, \qquad |z| < 1,$$

with $|\beta_k| < 1$, where the β_k are to be further restricted later. We then have[1]

[1] For the details here, the reader may refer to the writer's *Interpolation and approximation by rational functions in the complex domain*, Amer. Math. Soc. Colloquium Publications, vol. 20, New York, 1935, §10.1.

$$[z^\mu B_n(z), z^\mu] = [B_n(z), 1] = 4\pi[1 - |\beta_1\beta_2 \cdots \beta_n|].$$

If the β_k are chosen infinite in number so that the corresponding Blaschke product

$$B(z) \equiv \lim_{n\to\infty} B_n(z)$$

converges ($|z| < 1$), the sequence $B_n(z)$ converges (loc. cit.) in the mean on C to the boundary values of $B(z)$, and we have with $\psi(z) \equiv z^\mu B(z)$

$$[\psi(z), z^\mu] = 4\pi[1 - |\beta_1\beta_2 \cdots |];$$

the second member is less than ϵ if the β_k are suitably chosen.

2. **Main theorems.** We turn to an analogue of Rouché's theorem for $d(z) \equiv 1$.

THEOREM 2. *Let $f(z)$ be of class H_2, with*

$$[f(z), 1] < (1 - r^2), \qquad 0 < r < 1; \tag{5}$$

then $f(z)$ has no zeros in the closed region $|z| \leqq r$.

We set $f(z) \equiv \sum_0^\infty a_n z^n$. By Cauchy's algebraic inequality we have for $|z| = r$

$$\begin{aligned} |f(z) - 1| &\leqq |a_0 - 1| + \sum_1^\infty |a_n z^n| \\ &\leqq \left[|a_0 - 1|^2 + \sum_1^\infty |a_n|^2\right]^{1/2} \cdot \left[\sum_0^\infty |z^n|^2\right]^{1/2}. \end{aligned}$$

Of course we have, by the orthogonality properties of the powers of z,

$$[f(z), 1] = \left[|a_0 - 1|^2 + \sum_1^\infty |a_n|^2\right],$$

whence on $|z| = r$

$$\left|\frac{f(z) - 1}{1}\right| < 1; \tag{6}$$

it follows that $f(z)$ has no zeros in $|z| \leqq r$.

The conclusion of Theorem 2 is not valid if we replace the second member of (5) by any larger number, for if we set $f(z) \equiv r(r-z)/(1-rz)$, which has a zero on the circle $|z| = r$, the first member of (5) is $(1-r^2)$.

Theorem 2 is essentially a limiting case of

THEOREM 3. *Let* μ *be a positive integer, and suppose for some* r $(0<r<1)$ *we have*

$$[f(z), z^\mu] < r^{2\mu}(1-r^2),$$

where $f(z)$ *is of class* H_2; *then* $f(z)$ *has precisely* μ *zeros in the region* $|z| < r$ *and no zero on the circle* $|z| = r$.

More explicitly, suppose we have

$$0 < \epsilon < \epsilon_\mu = \mu^\mu/(1+\mu)^{1+\mu},$$

and denote by r_1 *and* r_2 $(0 < r_1 < r_2 < 1)$ *zeros of the equation*

$$r^{2\mu}(1-r^2) = \epsilon. \tag{7}$$

If $f(z)$ *is of class* H_2, *and if we have*

$$[f(z), z^\mu] < \epsilon, \tag{8}$$

then $f(z)$ *has precisely* μ *zeros in the region* $|z| < r_1$ *and no zeros in the closed annulus* $r_1 \leqq |z| \leqq r_2$.

We prove the latter part of Theorem 3, which includes the former part. It follows from Descartes's rule of signs that equation (7) has no more than two positive roots r_1 and r_2; the first member of (7) vanishes for $r=0$ and $r=1$, is positive in the interval $0<r<1$, and has there the maximum value ϵ_μ. The first member of (7) is greater than ϵ in the interval $r_1<r<r_2$.

If we set $f(z) \equiv \sum_0^\infty a_n z^n$, we have for $|z| = r < 1$ by Cauchy's inequality

$$|f(z) - z^\mu| \leqq [|a_0|^2 + |a_1|^2 + \cdots + |a_{\mu-1}|^2 + |a_\mu - 1|^2 + |a_{\mu+1}|^2 + \cdots]^{1/2} \left[\sum_0^\infty |z^n|^2\right]^{1/2}.$$

We also have

$$[f(z), z^\mu] = [|a_0|^2 + |a_1|^2 + \cdots + |a_\mu - 1|^2 + \cdots],$$

whence on $|z| = r$ (<1)

$$\left|\frac{f(z) - z^\mu}{z^\mu}\right| < \frac{\epsilon^{1/2}}{r^\mu(1-r^2)^{1/2}},$$

and this last member is not greater than unity for r in the closed interval $r_1 \leqq r \leqq r_2$. The conclusion of Theorem 3 follows from Rouché's theorem.

The latter part of Theorem 3 is not valid if we replace the second member of (8) by any larger number, for if we set $f(z)=z^\mu - r^\mu(1-r^2)/(1-rz)$, the function $f(z)$ has a zero $z=r$ on the circle $|z|=r$; for this function the first member of (8) is $r^{2\mu}(1-r^2)$.

At least so far as concerns approximation on C to the functions z^μ, $\mu \geqq 0$, Theorems 1, 2, and 3 give a complete solution to the problem proposed, namely the investigation of (2) as a measure of approximation, with reference to the number and location of the zeros of the approximating functions interior to C. These results have been established by the use of Rouché's theorem itself and standard methods; no new principle to replace Rouché's theorem is needed here.

The application of Theorem 2 in the study of a specific function $F(z)$ of class H_2 is not unique, for we may set $f(z) \equiv AF(z)$, where A is an arbitrary constant. It is natural to choose A so that $[AF(z), 1]$ is as small as possible; thus if we have $F(z)=\sum_0^\infty a_n z^n$, we should minimize

$$[AF(z), 1] = |Aa_0 - 1|^2 + |Aa_1|^2 + |Aa_2|^2 + \cdots .$$

It is clear that for given $|A|$ we should choose arg A so that Aa_0 is positive (we ignore the trivial case $a_0=0$), so we have $|Aa_0-1| = ||A||a_0|-1|$. The minimum for all $|A|$ of the function

$$||A||a_0| - 1|^2 + |A|^2|a_1|^2 + |A|^2|a_2|^2 + \cdots$$

occurs for $|A| = |a_0|/[|a_0|^2+|a_1|^2+ \cdots]$ and equals

$$1-|a_0|^2/[|a_0|^2+|a_1|^2+ \cdots].$$

Inequality (5) then takes the form $r^2<|a_0|^2/[|a_0|^2+|a_1|^2+ \cdots]$. It follows that *if the function $F(z)\equiv \sum_0^\infty a_n z^n$ with $a_0\neq 0$ is of class H_2, then $F(z)$ has no zero in the region*

$$|z| < |a_0|/[|a_0|^2 + |a_1|^2 + \cdots]^{1/2}.$$

This result is due to Petrovitch,[2] and was later studied also by Landau.[3]

Just as there are various ways of applying Theorem 2 to a specific function $F(z)$ of class H_2, there are various ways of applying Theorem 3. The minimum for all A of $[AF(z), z^\mu]$ is $1-|a_\mu|^2/[|a_0|^2+|a_1|^2 + \cdots]$. It follows (notation of Theorem 3) that *if the function $F(z)$ is of class H_2, and if we have*

[2] M. Petrovitch, Bull. Soc. Math. France vol. 29 (1901) pp. 303–312.

[3] E. Landau, Tôhoku Math. J. vol. 5 (1914) pp. 97–116.

$$1 - |a_\mu|^2/[|a_0|^2 + |a_1|^2 + \cdots] < \epsilon < \epsilon_\mu,$$

then $f(z)$ has precisely μ zeros in the region $|z| < r_1$ and no zeros in the closed annulus $r_1 \leqq |z| \leqq r_2$.

Theorem 3 is closely analogous to the well known theorem of Pellet, that the condition

$$|a_k| r^k > |a_0| + |a_1| r + \cdots + |a_{k-1}| r^{k-1} + |a_{k+1}| r^{k+1} + \cdots + |a_n| r^n$$

implies that the polynomial $\sum_0^n a_j z^j$ has precisely k zeros in the region $|z| < r$; Pellet's theorem applies also to a power series converging uniformly for $|z| = r$.

3. **Extremal functions.** For the sake of completeness we determine the extremal functions:

THEOREM 4. *If the hypothesis of Theorem 2 is modified by replacing the sign $<$ in (5) by the sign $\leqq$, then either $f(z)$ has no zeros in the closed region $|z| \leqq r$ or $f(z)$ is of the form $1-(1-r^2)/(1-\gamma r z)$ with $|\gamma| = 1$.*

If the hypothesis of Theorem 3 is modified by replacing the sign $<$ in (8) by the sign $\leqq$, then either $f(z)$ has precisely μ zeros in the region $|z| < r_1$ and no zeros in the closed annulus $r_1 \leqq |z| \leqq r_2$, or $f(z)$ is of the form $z^\mu - r_j^\mu(1-r_j^2)/\gamma^\mu(1-\gamma r_j z)$ with $|\gamma| = 1$, $j = 1$ or 2.

To establish the first part of Theorem 4, we merely notice that the original proof of Theorem 2 (in particular the use of Cauchy's inequality) remains valid under the modified hypothesis unless the two sets of numbers

$$a_0 - 1,\ a_1,\ a_2, \cdots, \quad 1,\ z,\ z^2, \cdots$$

are each proportional to the conjugate of the other, for some $z = z_0$, $|z_0| = r$:

$$a_0 - 1 = \lambda, \qquad a_1 = \lambda \bar{z}_0, \qquad a_2 = \lambda \bar{z}_0^2, \cdots.$$

Here we have (in any case under Theorem 4 for which the conclusion of Theorem 2 is not satisfied)

$$|a_0 - 1|^2 + \sum_1^\infty |a_n|^2 = 1 - r^2,$$

$$|\lambda|^2 \sum_0^\infty |z_0|^{2n} = 1 - r^2,$$

$$|\lambda| = 1 - r^2.$$

Moreover we have ($|z|<1$)

$$f(z) \equiv (1+\lambda) + \lambda\bar{z}_0 z + \lambda\bar{z}_0^2 z^2 + \cdots \equiv 1 + \lambda/(1-\bar{z}_0 z),$$

whose only zero is $z=(1+\lambda)/\bar{z}_0$; the modulus of this zero is $|1+\lambda|/r$, which is not less than $(1-|\lambda|)/r=r$ and is equal to r when and only when we have $\lambda=-(1-r^2)$.

The latter part of Theorem 4 is similarly proved. In any case under Theorem 4 not included in the original hypothesis, the original proof is valid unless we have for some $z=z_0$, $|z_0|=r_j$, $j=1$ or 2,

$$a_0=\lambda,\ a_1=\lambda\bar{z}_0,\ \cdots,\ a_{\mu-1}=\lambda\bar{z}_0^{\mu-1},\ a_\mu-1=\lambda\bar{z}_0^{\mu},\ a_{\mu+1}=\lambda\bar{z}_0^{\mu+1},\ \cdots.$$

Thus we have in any exceptional case

$$|a_0|^2+|a_1|^2+\cdots+|a_\mu-1|^2+|a_{\mu+1}|^2+\cdots=r_j^{2\mu}(1-r_j^2),$$

$$|\lambda|^2\sum_0^\infty |z_0|^{2n}=r_j^{2\mu}(1-r_j^2),$$

$$|\lambda|=r_j^\mu(1-r_j^2).$$

Moreover we have ($|z|<1$)

$$\begin{aligned} f(z) &\equiv \lambda+\lambda\bar{z}_0 z+\cdots+\lambda\bar{z}_0^{\mu-1}z^{\mu-1}+(1+\lambda\bar{z}_0^\mu)z^\mu \\ &\quad + \lambda\bar{z}_0^{\mu+1}z^{\mu+1}+\cdots \\ &\equiv z^\mu+\lambda/(1-\bar{z}_0 z). \end{aligned}$$

The zeros of $f(z)$ are the zeros of $\bar{z}_0 z^{\mu+1}-z^\mu-\lambda$, and in such a zero we have $|\lambda|=r_j^\mu(1-r_j^2)=|\bar{z}_0 z^{\mu+1}-z^\mu|=|z^\mu|\cdot|1-\bar{z}_0 z|$. Thus z is not a zero of $f(z)$ on $|z|=r_j$ unless we have $z=z_0$, $\lambda=-z_0^\mu\cdot(1-r_j^2)$; Theorem 4 is established.

We mention a further limiting case under Theorem 3: *If $f(z)$ is of class H_2 and we have $[f(z), z^\mu]=\epsilon_\mu$, then either $f(z)$ is of the form $z^\mu-r^\mu(1-r^2)/\gamma^\mu(1-\gamma r z)$ with $r=r_0=[\mu/(1+\mu)]^{1/2}$, $|\gamma|=1$, or $f(z)$ has precisely μ zeros in the region $|z|<r_0$ and no zeros on the circle $|z|=|r_0$.*

4. **Polynomials.** The methods already used apply also in the study of zeros of *polynomials of given degree ν*, namely functions of the form $p(z)\equiv\sum_0^\nu a_n z^n$. Here the circle $|z|=1$ is of no especial significance, but we continue to use the measure of approximation ($\nu\geqq\mu\geqq 0$)

$$\begin{aligned}[p(z), z^\mu] = |a_0|^2+|a_1|^2+\cdots \\ +|a_{\mu-1}|^2+|a_\mu-1|^2+|a_{\mu+1}|^2+\cdots+|a_\nu|^2.\end{aligned}$$

The analogue of Theorem 2 is

THEOREM 5. *Suppose* $p(z) \equiv \sum_0^\nu a_n z^n$ *with*

$$[p(z), 1] < 1/(1 + r^2 + r^4 + \cdots + r^{2\nu});$$

then $p(z)$ *has no zeros in the closed region* $|z| \leqq r$.

Theorem 5 is established by the same method as is Theorem 2; the proof is omitted. The analogue of Theorem 3 is

THEOREM 6. *Suppose* $p(z) \equiv \sum_0^\nu a_n z^n$, *suppose* $\mu(<\nu)$ *is a positive integer, and suppose*

$$[p(z), z^\mu] = A_\mu < r^{2\mu}/(1 + r^2 + r^4 + \cdots + r^{2\nu}); \tag{9}$$

then $p(z)$ *has precisely* μ *zeros in the region* $|z| < r$ *and no zeros on the circle* $|z| = r$.

Consequently, if the equation

$$1 + r^2 + r^4 + \cdots + r^{2\nu} - r^{2\mu}/A_\mu = 0, \qquad A_\mu \neq 0, \tag{10}$$

has two positive zeros r_1 *and* r_2 $(>r_1)$, *then* $p(z)$ *has no zeros in the annulus* $r_1 < |z| < r_2$, *and has precisely* μ *zeros in the closed region* $|z| \leqq r_1$.

It follows by Descartes's rule of signs that (10) has no more than two positive zeros. Moreover the first member of (10) is positive for $r = 0$ and $r \to +\infty$, so if (10) has two positive zeros as indicated, inequality (9) is satisfied in the interval $r_1 < r < r_2$. Theorem 6 follows by the Cauchy inequality

$$\begin{aligned} | a_0 + a_1 z + \cdots + (a_\mu - 1)z^\mu + \cdots + a_\nu z^\nu |^2 \\ \leqq A_\mu(1 + r^2 + \cdots + r^{2\nu}), \end{aligned}$$

whence we have on the circle $|z| = r$, $r_1 < r < r_2$,

$$\left| \frac{p(z) - z^\mu}{z^\mu} \right| < 1,$$

and by Rouché's theorem.

A further result for polynomials, which gives an upper bound for the moduli of the zeros, and which has no analogue for arbitrary functions of class H_2, is

THEOREM 7. *Suppose* $p(z) \equiv \sum_0^\nu a_n z^n$ *and suppose*

$$[p(z), z^\nu] = A_\nu < r^{2\nu}/(1 + r^2 + \cdots + r^{2\nu}); \tag{11}$$

then all zeros of $p(z)$ *lie in the region* $|z| < r$.

The proof of Theorem 7 is similar to that of Theorem 6, and can also be given from Theorem 5 by the substitution $w=1/z$; it is left to the reader.

We have already indicated that it may be more favorable to apply Theorems 2 and 3 to the function $AF(z)$ rather than to a given function $F(z)$. A similar remark applies to Theorems 5–7. We formulate:

If we have $p(z)\equiv\sum_0^\nu a_nz^n$ *with*

$$1-|a_0|^2/[|a_0|^2+|a_1|^2+\cdots+|a_\nu|^2] < 1/(1+r^2+r^4+\cdots+r^{2\nu}),$$

then $p(z)$ *has no zeros in the closed region* $|z|\leqq r$.

If we have $p(z)=\sum_0^\nu a_nz^n$, $0<\mu<\nu$, *with*

$$1-|a_\mu|^2/[|a_0|^2+|a_1|^2+\cdots+|a_\nu|^2] < r^{2\mu}/(1+r^2+r^4+\cdots+r^{2\nu}),$$

then $p(z)$ *has precisely* μ *zeros in the region* $|z|<r$ *and no zero on the circle* $|z|=r$. *Consequently if the equation*

$$1+r^2+r^4+\cdots+r^{2\nu}-r^{2\mu}[|a_0|^2+|a_1|^2+\cdots +|a_\nu|^2]/[|a_0|^2+\cdots+|a_{\mu-1}|^2+|a_{\mu+1}|^2+\cdots+|a_\nu|^2]=0$$

has two positive zeros r_1 *and* r_2 $(>r_1)$, *then* $p(z)$ *has no zeros in the annulus* $r_1<|z|<r_2$ *and has precisely* μ *zeros in the closed region* $|z|\leqq r_1$.

If we have $p(z)\equiv\sum_0^\nu a_nz^n$, *with*

$$1-|a_\nu|^2/[|a_0|^2+|a_1|^2+\cdots+|a_\nu|^2] < r^{2\nu}/(1+r^2+r^4+\cdots+r^{2\nu}),$$

then all zeros of $p(z)$ *lie in the region* $|z|<r$.

Theorems 5–7 contain bounds which cannot be improved. For the sake of completeness we determine the extremal functions:

THEOREM 8. *If the hypothesis of Theorem* 5 *is modified by replacing the sign* $<$ *by the sign* $\leqq$, *then either* $p(z)$ *has no zeros in the closed region* $|z|<r$ *or we have* $p(z)\equiv 1-(1-r^2)(1-\gamma^{\nu+1}r^{\nu+1}z^{\nu+1})/(1-r^{2\nu+2})(1-\gamma rz)$, *with* $|\gamma|=1$.

If the hypothesis of Theorem 6 *is modified by replacing in* (9) *the sign* $<$ *by the sign* $\leqq$, *then either* $p(z)$ *has precisely* μ *zeros in the closed region* $|z|\leqq r$ *or we have*

$$p(z)\equiv z^\mu-r^\mu(1-r^2)(1-\gamma^{\nu+1}r^{\nu+1}z^{\nu+1})/\gamma^\mu(1-r^{2\nu+2})(1-\gamma rz)$$

with $|\gamma|=1$. *Consequently if* (10) *has two positive zeros* r_1 *and* r_2, *then either* $p(z)$ *has no zeros in the closed annulus* $r_1\leqq|z|\leqq r_2$ *and has pre-*

cisely μ *zeros in the closed region* $|z| \leqq r_1$, *or* $p(z)$ *is of the form*

$$z^{\mu} - r_j^{\mu}(1 - r_j^2)(1 - \gamma^{\nu+1} r_j^{\nu+1} z^{\nu+1})/\gamma^{\mu}(1 - r_j^{2\nu+2})(1 - \gamma r_j z), \quad j = 1 \text{ or } 2.$$

If the hypothesis of Theorem 7 *is modified by replacing in* (11) *the sign* $<$ *by the sign* $\leqq$, *then either* $p(z)$ *has all its zeros in the region* $|z| < r$ *or we have* $p(z) \equiv z^{\nu} - r^{\nu}(1-r^2)(1-\gamma^{\nu+1}r^{\nu+1}z^{\nu+1})/\gamma^{\nu}(1-\gamma r z) \cdot (1-r^{2\nu+2})$ *with* $|\gamma| = 1$.

5. **Related problems.** Theorems 2–4 can properly be viewed as analogues of Rouché's theorem, for approximation on C to the functions z^{μ}, insofar as such analogues exist; sufficient conditions are derived that $f(z)$ and z^{μ} should have *the same number of zeros* in a suitable region interior to C. Still another problem suggests itself, however:

PROBLEM I. *To determine the smallest number* η_{μ} $(\mu > 0)$ *such that for a function* $f(z)$ *of class* H_2 *the inequality* $[f(z), z^{\mu}] < \eta_{\mu}$ *implies that* $f(z)$ *has at least one zero interior to* C.

Problem I is still open, but it is clear that from Theorem 3 we have $\eta_{\mu} \geqq \mu^{\mu}/(1+\mu)^{1+\mu}$. Moreover for the specific function

$$f_0(z) \equiv -(z^{\mu} - 1)^2(z^{\mu} + 4)/10,$$

which has no zeros interior to C, we have $[f_0(z), z^{\mu}] = 3/10$, whence $\eta_{\mu} \leqq 3/10$.

Modifications of Problem I suggest themselves:

PROBLEM II. *To determine the smallest number* $\eta_{\mu}^{(\beta)}$ *such that for a function* $f(z)$ *of class* H_2 *the inequality* $[f(z), z^{\mu}] < \eta_{\mu}^{(\beta)}$, $\mu \geqq \beta > 0$, *implies that* $f(z)$ *has at least* β *zeros interior to* C.

PROBLEM III. *To determine the smallest number* $\eta_{\mu}^{(\beta)}(\nu)$ *such that for a polynomial* $p(z)$ *of degree* ν *the inequality*

$$[p(z), z^{\mu}] < \eta_{\mu}^{(\beta)}(\nu), \qquad \nu \geqq \mu \geqq \beta > 0,$$

implies that $p(z)$ *has at least* β *zeros interior to* C.

A further obvious problem is to replace the metric $[f(z), F(z)]$ by the new metric

$$\frac{1}{2\pi}\int_C |f(z) - F(z)|^p |dz|, \qquad p > 0;$$

Cauchy's integral formula then provides a bound on $|f(z) - F(z)|$ on the circle $|z| = r < 1$, but this new metric has no simple relation to the coefficients in the Taylor developments of $f(z)$ and $F(z)$.

HARVARD UNIVERSITY