SOME THEOREMS ON MEROMORPHIC FUNCTIONS

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1. Introduction. In a recent paper [1] Yoshitomo Okada proved the following two theorems.

**Theorem A.** If for any meromorphic function

\[ F(z) = f(z)/g(z), \]

where \( f \) and \( g \) are canonical products of genera \( p, q \) and of orders \( \rho_1, \rho_2 \) respectively,

\( \max (\rho_1, \rho_2) = \max (\rho, q), \)

then

\[
\liminf_{r \to \infty} \frac{1}{rN(r)\phi(r)} \int_0^r \log^+ M(t, F)dt = 0,
\]

where \( N(r) = n(r, f) + n(r, g) \) and \( \phi(x) \) is any positive continuous non-decreasing function of a real variable \( x \) such that \( \int_1^x dx/x\phi(x) \) is convergent.

**Theorem B.** If (1) is a function of order \( \rho \), where \( \rho > 0 \) is not an integer, then

\[
\liminf_{r \to \infty} \frac{1}{rN(r)} \int_0^r \log^+ M(t, F)dt < \infty.
\]

In this paper we extend Theorems A and B. Let

\[ F(z) = z^k \exp(H(z))f(z)/g(z) \]

be any meromorphic function of finite order \( \rho \). Here \( H(z) \) is a polynomial of degree \( h \); \( f(z) \) and \( g(z) \) are canonical products of orders \( \rho_1, \rho_2 \) and genera \( p, q \) respectively. The genus of \( F(z) \) is \( P = \max (\rho, q, h) \) and we have \( \rho - 1 \leq P \leq \rho \). Let \( n(r, 0) \) and \( n(r, \infty) \) denote the number of zeros and poles respectively of \( F(z) \) in \( |z| \leq r \) and write \( \psi(r) = n(r, f) + n(r, g) \),

\[ I(r, F) = I(r) = \frac{1}{r} \int_0^r \log^+ M(t, F)dt. \]

**Theorem 1.** If for any meromorphic function (5) of order \( \rho \) where

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1 Numbers in brackets refer to the references at the end of the paper.
\( p > 0 \) is an integer,

\[ h \leq \max (p, q) = s \text{ (say)}, \]

then

\[ \liminf_{r \to \infty} \frac{I(r, F)}{\{n(r, 0) + n(r, \infty)\} \phi(r)} = 0, \]

where \( \phi(x) \) has been defined in the statement of Theorem A.

**Theorem 2.** For any meromorphic function \( (5) \) of order \( p \) where \( p > 0 \) is not an integer, we have

\[ \liminf_{r \to \infty} \frac{I(r, F)}{\{n(r, 0) + n(r, \infty)\} < \infty. \]

**Theorem 3.** If the meromorphic function \( (5) \) be nonconstant and of zero order, then

\[ \liminf_{r \to \infty} \frac{I(r, F)}{\{N(r, 0) + N(r, \infty)\} < \infty, \]

where \( N(r, a) \) denotes as usual

\[ \int_{0}^{r} \frac{n(x, a) - n(0, a)}{x} \, dx + n(0, a) \log r. \]

**Corollary.**

\[ \liminf_{r \to \infty} \frac{I(r, F)}{\{n(r, 0) + n(r, \infty)\} \log r < \infty. \]

2. **Examples.** If \( h > \max (p, q) \), then (7) does not hold. For instance, if

\[ F(z) = e^{z} \prod_{1}^{\infty} \left( 1 + \frac{z}{n(\log n)^{\alpha}} \right), \quad \alpha > 1, \]

then \( h = 1, \max (p, q) = 0, \) and

\[ \frac{I(r)}{\{n(r, 0) + n(r, \infty)\} \log r \log \log r^{2}} \to \infty, \]

as \( r \to \infty. \) Further let \( \alpha(x) \) be any given function tending to infinity, however slowly, with \( x \) and consider

\[ F(z) = \prod_{1}^{\infty} E\left( \frac{z}{\alpha}, \ p \right), \]

where
\[ \alpha_\gamma = -\left\{ \nu (\log \nu)^\gamma \right\}^{1/\gamma}, \quad \rho < \gamma < \rho + 1; \]

then \( F(z) \) is an entire function of nonintegral order \( \rho > 0 \) and we have [5, p. 44]

\[
\lim_{r \to \infty} \frac{I(r, F)\alpha(r)}{\{n(r, 0) + n(r, \infty)\}} = \infty.
\]

Further \( F(z) = \prod_{n=1}^\infty (1 - \frac{z}{e^n}) \) is an entire function of zero order for which

\[
\lim_{r \to \infty} \frac{I(r, F)\alpha(r)}{\{N(r, 0) + N(r, \infty)\}} = \infty,
\]

\[
\lim_{r \to \infty} \frac{I(r, F)}{\{n(r, 0) + n(r, \infty)\} \log r} = \frac{1}{2}.
\]

**3. Lemma.** Let

\[
J(r, \rho) = \int_0^\infty \frac{r^{\rho+1}\psi(t)dt}{t^{\rho+1}(t + r)}.
\]

If \( s \geq h \) and \( \psi(t) \geq 1 \) for all large \( t \), then

\[\begin{equation}
I(r, F) < HJ(r, s). \tag{11}
\end{equation}\]

**Proof.** Let \((a_n)_n^\infty\) denote the zeros of \( f(z) \) and \((b_n)_n^\infty\) the zeros of \( g(z) \) and let \( k > 1 \). Then

\[
T(r, F) < T(r, f) + T(r, g) + O(r^h + \log r).
\]

\[
I(r, F) < HT(kr, F) < H\left\{ T(kr, f) + T(kr, g) \right\} + O(r^h + \log r)
\]

\[
< H\left\{ \log^+ M(kr, f) + \log^+ M(kr, g) \right\} + O(r^h + \log r)
\]

\[
< H\left\{ \sum_{n=1}^\infty \frac{r^{\rho+1}}{|a_n|^\gamma(r + |a_n|)} + \sum_{n=1}^\infty \frac{r^{\rho+1}}{|b_n|^\gamma(r + |b_n|)} \right\}
\]

\[+ O(r^h + \log r)
\]

\[
< H \int_0^\infty \frac{r^{\rho+1}\psi(t)dt}{t^{\rho+1}(t + r)} + O(r^h + \log r).
\]

Now

\[
J(r, s) > \frac{1}{2} \int_0^r \frac{r^s\psi(t)dt}{t^{s+1}} > hr^s.
\]

Hence if \( s > 0 \), \( I(r) < HJ(r, s) \). If \( s = 0 \) then \( h = 0 \) and since \( \psi(t) \geq 1 \) for all large \( t \),

\[\begin{equation}
^1 H(h) \text{ denotes a positive constant not necessarily the same at each occurrence.}
\end{equation}\]
which proves the lemma.

4. Proof of Theorem 1. We note that \( \psi(r) \leq 1 \) for all large \( r \), for if \( \psi(r) = 0 \) for all \( r \), then \( s = 0 \) and hence, by (6), \( h = 0 \) and \( F(z) \) would then not be a function of order greater than or equal to one.

Consider \( G(z) = \prod E(z/c_n, s) \), where the sequence \( c_1, c_2, \ldots \) is composed of \( a_1, a_2, \ldots, b_1, b_2, \ldots \) and \( |c_1| \leq |c_2| \leq \ldots \). Since \( h \leq s \), \( G(z) \) is an entire function of order \( p \) and genus \( s = \max (p, q) \). Further \( s = \rho \) or \( \rho - 1 \) and hence we have \([2, pp. 23–29; 3, pp. 180–186]\)

\[
\lim_{r \to \infty} \frac{J(r, s)}{\psi(r) \phi(r)} = 0.
\]

Since
\[
\psi(r) \leq n(r, 0) + n(r, \infty),
\]
(7) follows from the lemma.

5. Proof of Theorem 2. This theorem follows from the argument of Okada \([1, p. 249]\). We sketch an alternative proof. Let \( \{p\} = P \). Then \( h \leq P = s \). Let \( 0 < \epsilon < \min \{p - s, s + 1 - p\} \). From the lemma we have

\[
I(r, F) < H \left\{ \int_0^r \frac{\psi(t) dt}{t^{\rho+1}} + r^{\rho+1} \int_r^\infty \frac{\psi(t) dt}{t^{\rho+2}} \right\}.
\]

From Lemma 3 \([3, p. 184]\) we have

\[
\frac{\psi(t)}{t^{\rho+1}} \leq \frac{\psi(r_n)}{r_n^{\rho+1}}, \quad 0 \leq t \leq r_n; \quad \frac{\psi(t)}{t^{\rho+2}} \leq \frac{\psi(r_n)}{r_n^{\rho+2}}, \quad t \geq r_n,
\]
for a sequence \( (r_n)_1 \). \( r_n \uparrow \infty \). Hence the theorem follows.

6. Proof of Theorem 3. If \( \psi(t) \equiv 0 \) for all \( t \) then \( F(z) \) would be of the form \( Az^k \) and (9) and (10) obviously hold. Hence we may suppose that \( \psi(t) \geq 1 \) for all large \( t \). Let

\[
\psi_1(t) = \int_0^r \frac{\psi(t)}{t} dt.
\]

Then

\[
I(r) < H \left[ \int_0^r \frac{\psi(t)}{t} dt + r \int_r^\infty \frac{\psi(t)}{t^2} dt \right] = Hr \int_r^\infty \frac{\psi_1(t)}{t} dt.
\]
Now \[ \lim_{{r \to \infty}} \frac{\psi_1(r)}{r^s} = 0. \]

Hence there exists a sequence \( \{r_n\}_{n=1}^\infty, r_n \uparrow \infty \), such that
\[ \frac{\psi_1(r)}{r^s} \leq \frac{\psi_1(r_n)}{r_n^s} \]
for \( r \geq r_n \);
and the theorem follows. The corollary follows directly from the theorem.

**References**


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