A THEOREM FOR KERNEL FUNCTIONS

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Let B be a domain lying in the complex z plane and $K_B(z, \bar{t})$ its kernel function. A number of relationships exist between the kernel and the geometric properties of the domain. (See, for example, [1].)[1] It is the purpose of the present note to relate the successive derivatives of the kernel with the domain B.

If z is interior to B, we shall denote by $r_B(z)$ the shortest distance from the point z to the boundary of B. Furthermore, we introduce the abbreviation

$$K_B^{(m,n)}(z, \bar{t}) \equiv \frac{\partial^{m+n}}{\partial z^m \partial \bar{t}^n} K_B(z, \bar{t}). \tag{1}$$

THEOREM 1. *Let the domain B be such that at every boundary point z_1 there exists*[2] *a circle exterior to B and passing through z_1. Then,*

$$1/r_B(z) = \limsup_{n\to\infty} (e/n)\left[K_B^{(n,n)}(z, \bar{z})\right]^{1/2n}. \tag{2}$$

PROOF. It obviously suffices to prove the theorem for $z=0$, it being supposed that B contains this point in its interior.

Let $r=r_B(0)$, and suppose that z_1 is a point which simultaneously lies on the boundary of B and on the circle $|z|=r$. In the bicylinder $|z| \leqq r^* < r$, $|t| \leqq r^* < r$, $K_B(z, \bar{t})$ is an analytic function of the two complex variables z and $\bar{t}$ and hence has an expansion of the form

$$K_B(z, \bar{t}) = \sum_{m,n=0}^{\infty} K_B^{(m,n)}(0, 0) z^m \bar{t}^n / m!n! \tag{3}$$

converging absolutely and uniformly there. If, then, λ is a real variable with $|\lambda| < r/r^*$, then

$$K_B(\lambda z, \lambda \bar{t}) = \sum_{m,n=0}^{\infty} K_B^{(m,n)}(0, 0) z^m \bar{t}^n \lambda^{m+n} / m!n!. \tag{4}$$

With λ in the above range, (4) converges absolutely and uniformly in $|z|$, $|t| \leqq r^*$. The circle $|z| = r^*$ will be designated by C^*. Thus, for $|\lambda| < r/r^*$,

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[1] Number 1 in brackets refers to the bibliography at the end of the paper.

[2] The theorem may be established for a wider class of boundaries. Moreover, a similar result holds for several complex variables.

$$
\begin{aligned}
F(\lambda) &\equiv \iint_{C^*} K_B(\lambda z, \lambda \bar{z})dxdy \\
&= \sum_{m,n=0}^{\infty} K_B^{(m,n)}(0, 0)\lambda^{m+n}/m!n! \iint_{C^*} z^m \bar{z}^n dxdy \qquad (5)\\
&= \sum_{n=0}^{\infty} K_B^{(n,n)}(0, 0)r^{*2n+2}\lambda^{2n}/(n!)(n+1)!.
\end{aligned}
$$

The function $F(\lambda)$ is therefore analytic for $|\lambda| < r/r^*$.

We shall prove, moreover, that $F(\lambda)$ has a singularity at the point $\lambda = r/r^*$. If this is admitted momentarily, then by the familiar Cauchy-Hadamard formula for the radius of convergence of a power series, we shall have

$$
\begin{aligned}
r^*/r &= \limsup_{n\to\infty} \left[K_B^{(n,n)}(0, 0)r^{*2n+2}/n!(n+1)!\right]^{1/2n} \qquad (6)\\
&= r^* \limsup_{n\to\infty} (e/n)\left[K_B^{(n,n)}(0, 0)\right]^{1/2n}
\end{aligned}
$$

from which (2) follows.

We shall show that

$$
\lim_{\lambda\to r/r^*-} F(\lambda) = \infty. \qquad (7)
$$

Suppose that C_1 designates a circle of radius r_1 which is exterior to B but whose circumference passes through z_1. Designate the exterior of the circle by D. By the monotonicity of the kernel function

$$
K_B(\lambda z, \lambda \bar{z}) \geqq K_D(\lambda z, \lambda \bar{z}). \qquad (8)
$$

Therefore,

$$
\begin{aligned}
F(\lambda) &\geqq \iint_{C^*} K_D(\lambda z, \lambda \bar{z})dxdy \qquad (9)\\
&= \lambda^{-2} \iint_{\lambda C^*} K_D(z, \bar{z})dxdy \equiv I(\lambda).
\end{aligned}
$$

In the above equation, λC^* designates the circle $|z| \leqq \lambda r^*$. To estimate the integral $I(\lambda)$, it is convenient to introduce new coordinates $z' = x' + iy'$ as follows: z_1 shall be the new origin, and the center of C_1 shall lie on the negative x' axis. In the new system, K_D is given by

$$
K_D(z', \bar{z}') = \frac{1}{\pi}\left(z' + \bar{z}' + \frac{z'\bar{z}'}{r_1}\right)^{-2} = \frac{1}{\pi}\left(2x' + \frac{x'^2 + y'^2}{r_1}\right)^{-2}. \qquad (10)
$$

For values of λ sufficiently near to r/r^*, the ray $x'=y'$ will intersect the circle λC^* in two points whose abscissas $x_1'(\lambda)$, $x_2'(\lambda)$ have the property

$$\lim_{\lambda\to r/r^*-} x_1'(\lambda) = 0; \qquad \lim_{\lambda\to r/r^*-} x_2'(\lambda) = r. \tag{11}$$

Designate by $T(\lambda)$ the trapezoid bounded by $y'=0$, $y'=x'$, $x'=x_i'(\lambda)$ $(i=1, 2)$. $T(\lambda)$ is contained in the circle λC^*. Now,

$$x'^2K_D(z', \bar{z}') = \frac{1}{\pi}(2 + x'/r_1 + y'^2/x'r_1)^{-2} \tag{12}$$

so that if (x', y') is confined to $T(\lambda)$, then $0 \leqq x' \leqq r$, $0 \leqq y' \leqq r$, $0 \leqq y'/x' \leqq 1$, and hence

$$x'^2K_D(z', \bar{z}') \geqq \frac{1}{4\pi}(1 + (r/r_1))^{-2} = k > 0 \tag{13}$$

for z' in $T(\lambda)$. Combining our inequalities we have

$$I(\lambda) \geqq \lambda^{-2}\iint_{T(\lambda)} k/x'^2dx'dy' = \lambda^{-2}\log\left[x_2'(\lambda)/x_1'(\lambda)\right]. \tag{14}$$

The assertion (7) now follows from (9) and (11), and the proof is complete.

As an application of Theorem 1, we show how, for the case of a simply-connected domain, the quantity $r_B(0)$ may be expressed in terms of the coefficients of the mapping function of the domain onto the unit circle.

THEOREM 2. *Let $w(z)=a_1z+a_2z^2+\cdots$ map the simply connected domain B onto the unit circle in the w-plane, $z=0$ corresponding to $w=0$. Then,*

$$1/r = \limsup_{n\to\infty}\left\{\sum_{\nu=0}^{n}(\nu+1)\left|\sum_{j=\nu}^{n}\frac{(n-j+1)}{j!}a_{n-j+1}\left(\frac{d^jw^\nu}{dz^j}\right)_0\right|^2\right\}^{1/2n}. \tag{15}$$

PROOF. If C designates the unit circle in the w-plane, then by the conformal invariance of the kernel function we have

$$K_B(z, \bar{z}) = K_C(w, \bar{w})\frac{dw}{dz}\overline{\left(\frac{dw}{dz}\right)}.^3 \tag{16}$$

Now,

[3] $z^- = x - iy$.

$$(17)\qquad \frac{\partial^n}{\partial z^n} K_B(z, \bar{z}) = \sum_{j=0}^{n} C_{n,j} \overline{\left(\frac{dw}{dz}\right)} \frac{\partial^j K_C(w, \bar{w})}{\partial z^j} \frac{d^{n-j+1} w}{dz^{n-j+1}} .$$

Furthermore,

$$(18)\qquad \begin{aligned} &\frac{\partial^n}{\partial \bar{z}^n}\left(\overline{\left(\frac{dw}{dz}\right)} \frac{\partial^j K_C(w, \bar{w})}{\partial z^j}\right) \\ &\qquad = \sum_{p=0}^{n} C_{n,p} \frac{\partial^{j+p} K_C(w, \bar{w})}{\partial z^j \partial \bar{z}^p} \overline{\left(\frac{d^{n-p+1} w}{dz^{n-p+1}}\right)} \end{aligned}$$

so that

$$(19)\qquad \begin{aligned} &K_B^{(n,n)}(z, \bar{z}) \\ &\qquad = \sum_{j=0,\, p=0}^{n} C_{n,j} C_{n,p} \frac{d^{n-j+1} w}{dz^{n-j+1}} \overline{\left(\frac{d^{n-p+1} w}{dz^{n-p+1}}\right)} \frac{\partial^{j+p} K_C(w, \bar{w})}{\partial z^j \partial \bar{z}^p} . \end{aligned}$$

Again,

$$(20)\qquad K_C(w, \bar{w}) = \frac{1}{\pi} \sum_{\nu=0}^{\infty} (\nu + 1) w^\nu \bar{w}^\nu .$$

Thus

$$(21)\qquad \frac{\partial^{j+p}}{\partial z^j \partial \bar{z}^p} K_C(w, \bar{w}) = \frac{1}{\pi} \sum_{\nu=0}^{\infty} (\nu + 1) \frac{d^j w^\nu}{dz^j} \overline{\left(\frac{d^p w^\nu}{dz^p}\right)} .$$

Hence

$$\begin{aligned} &K_B^{(n,n)}(0, 0) \\ &\quad = \frac{1}{\pi} \sum_{j,p=0}^{n} C_{n,j} C_{n,p} \left(\frac{d^{n-j+1} w}{dz^{n-j+1}}\right)_0 \overline{\left(\frac{d^{n-p+1} w}{dz^{n-p+1}}\right)_0} \\ &\qquad\qquad \cdot \sum_{\nu=0}^{\min(p,j)} (\nu + 1) \left(\frac{d^j w^\nu}{dz^j}\right)_0 \overline{\left(\frac{d^p w^\nu}{dz^p}\right)_0} \\ &\quad = \frac{(n!)^2}{\pi} \sum_{\nu=0}^{n} (\nu + 1) \sum_{j,p=0}^{n} (n - j + 1)(n - p + 1) a_{n-j+1} \overline{(a_{n-p+1})} \\ &\qquad\qquad \cdot \left(\frac{1}{j!}\right)\left(\frac{1}{p!}\right)\left(\frac{d^j w^\nu}{dz^j}\right)_0 \overline{\left(\frac{d^p w^\nu}{dz^p}\right)_0} \\ &\quad = \frac{(n!)^2}{\pi} \sum_{\nu=0}^{n} (\nu + 1) \left| \sum_{j=\nu}^{n} \frac{(n - j + 1)}{j!} a_{n-j+1} \left(\frac{d^j w^\nu}{dz^j}\right)_0 \right|^2 . \end{aligned}$$

An application of Theorem 1 now yields (15).

Bibliography

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REMARK ON THE HURWITZ ZETA FUNCTION

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The Hurwitz zeta function, defined for $0<a\leqq 1$, $\Re(s)>1$, by

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(a+n)^s}, \tag{1}$$

is given [2][1] in the negative half-plane by means of

$$\begin{aligned} \zeta(1-s, a) &= \frac{2\Gamma(s)}{(2\pi)^s}\left(\cos\frac{\pi s}{2}\sum_{n=1}^{\infty}\frac{\cos(2\pi a n)}{n^s}\right. \\ &\qquad \left. + \sin\frac{\pi s}{2}\sum_{n=1}^{\infty}\frac{\sin(2\pi a n)}{n^s}\right) \\ &= \frac{2\Gamma(s)}{(2\pi)^s}\sum_{n=1}^{\infty}\frac{\cos(\pi s/2 - 2\pi a n)}{n^s} \qquad (\Re(s)>1). \end{aligned} \tag{2}$$

The functional equation for the Riemann zeta function is obtained from (2) upon setting $a=1$.

The more general function

$$\phi(x, a, s) = \sum_{n=0}^{\infty} \frac{e^{2\pi i n x}}{(a+n)^s} \tag{3}$$

reduces to $\zeta(s, a)$ when x is an integer. Lerch [1] derived the transformation formula

$$\begin{aligned} \phi(x, a, 1-s) = \frac{\Gamma(s)}{(2\pi)^s}\Big\{ & e^{\pi i(s/2 - 2ax)}\phi(-a, x, s) \\ & + e^{\pi i(-s/2 + 2a(1-x))}\phi(a, 1-x, s)\Big\}, \end{aligned} \tag{4}$$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.