SELF-ADJOINT FACTORIZATION OF DIFFERENTIAL OPERATORS

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In this short paper we prove the following result:

**Theorem.** Let \( L \) be an ordinary linear differential operator
\[
L = p_0(x) \frac{d^n}{dx^n} + p_1(x) \frac{d^{n-1}}{dx^{n-1}} + \cdots + p_n(x).
\]
of even order \( n = 2r \). \( p_i(x) \in C^{n-i} \) and \( p_0(x) > 0 \) in some closed finite interval \([a, b] \). Then there exists a subinterval of \([a, b]\) in which \( L \) has a factorization
\[
L = f(x) P_1 P_2 \cdots P_r
\]
where each \( P_a \) is of the second order and formally self-adjoint.

The theorem follows by complete induction after the proofs of

**Lemma 1.** Let
\[
N = d^n + q_1(x) \frac{d^{n-1}}{dx^{n-1}} + \cdots + q_n(x).
\]
be a linear differential operator with \( q_i(x) \in C^0 \) in some closed finite interval \([a, b]\). Then there is a subinterval \([a', b']\) of \([a, b]\) in which \( N \) has the representation
\[
N = PM
\]
where \( P = P^+ \) is of second order.

**Proof.** Let \( \{ \phi_1(x), \phi_2(x), \cdots, \phi_n(x) \} \) be \( n \) linearly independent solutions of \( Nu = 0 \) with Wronskian \( W(x) \). There exist \( n - 2 \) functions among the \( \phi_1, \phi_2, \cdots, \phi_n \) whose Wronskian \( \omega(x) \) is not identically zero in some subinterval of \([a, b]\). Let these \( n - 2 \) functions be \( \phi_1(x), \phi_2(x), \cdots, \phi_{n-2}(x) \) and let \( \omega(x) \) be unequal to zero in \([a', b']\).

Define the operator \( M \) by the equation:

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Let

\[ P = a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx} + c(x) \]

be so chosen that \( PM = N \). Now

\[ PMu = a s_2 u^{(n)} + [a(2s'_2 + ss) + bs_2] u^{(n-1)} + \cdots, \]
\[ Nu = u^{(n)} + q_1 u^{(n-1)} + \cdots. \]

Comparing coefficients and noting that \((Ws_2)'+(Ws_3)=0\), we see that \( a'(x) = b(x) \) and hence that \( P = P^+ \).

**Lemma 2.** Let

\[ L = \frac{d^n}{dx^n} + \frac{d^{n-1}}{dx^{n-1}} + \cdots + \frac{d}{dx} \]

be a linear differential operator with \( p_i(x) \in C^{n-i}, \ p_0(x)>0 \) in some closed finite interval \([a, b]\). Then there is subinterval \([a', b']\) of \([a, b]\) such that \( L \) has a representation

\[ L = SQ \]

where \( Q = Q^+ \) is of second order.

**Proof.** Let \( N = (1/p_0(x))L \). Then \( N \) is a linear differential operator with leading coefficient 1. Hence \( N^+ \) has leading coefficient 1. By Lemma 1, there exists a subinterval of \([a, b]\) such that

\[ N^+ = QR \]

with \( Q = Q^+ \). Taking adjoints of the above equation:

\[ N = R^+Q^+ = R^+Q. \]

Now

\[ L = p_0(x)N = p_0(x)R^+Q. \]

Let \( p_0(x)R^+ = S \). Then \( L = SQ \).