THE WEDDERBURN PRINCIPAL THEOREM
IN BANACH ALGEBRAS

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The Principal Theorem of Wedderburn for a finite-dimensional algebra \( A \) states that \( A \) is the vector space direct sum of its radical \( R \) and an algebra isomorphic to \( A/R \). It will be shown that the corresponding theorem is not true for all Banach algebras, but that it is true with certain restrictions.

The terminology of Jacobson [3] will be followed for radical, quasi-inverse, and quasi-regular. The notations \( x \odot y = x + y + xy \) and \( x' \) for the quasi-inverse of \( x \) will also be employed.

DEFINITION 1. A Banach algebra is a complete normed linear space which is also an algebra over the complex numbers satisfying \( \|xy\| \leq \|x\| \|y\| \). All the following results are proved for real algebras in [1] by the same methods.

To show that the Wedderburn theorem does not hold for an arbitrary Banach algebra, consider the commutative algebra \( A \) which is the completion of the algebra of all finite sums

\[
\sum_{i=1}^{n} \alpha_i e_i + \beta r
\]

where \( \alpha_i \) and \( \beta \) are complex, \( e_i \) are mutually orthogonal idempotents, \( r^2 = 0 \), \( e_i e_j = \delta_{ij} e_i \), and

\[
\| \sum \alpha_i e_i + \beta r \| = \max \left\{ \left( \sum |\alpha_i|^2 \right)^{1/2}, \ |\beta - \sum \alpha_i| \right\}.
\]

It is easy to show this defines a norm, but it is also necessary to verify that \( \|xy\| \leq \|x\| \|y\| \). Let \( x = \sum \alpha_i e_i + \gamma r \), \( y = \sum \beta_i e_i + \nu r \). Then

\[
x y = \sum \alpha_i \beta_i e_i; \quad \|xy\| = \max \left\{ \left( \sum |\alpha_i \beta_i|^2 \right)^{1/2}, \ \left( \sum \sum |\alpha_i \beta_i|^2 \right)^{1/2} \right\}.\]

By the Cauchy inequality,

\[
\sum |\alpha_i \beta_i| \leq \sum |\alpha_i| \beta_i \leq \left( \sum |\alpha_i|^2 \right)^{1/2} \left( \sum \beta_i^2 \right)^{1/2}.
\]

Together with \( \sum |\alpha_i \beta_i|^2 \leq \sum |\alpha_i|^2 \sum |\beta_i|^2 \) this shows \( \|xy\| \leq \|x\| \|y\| \). Hence \( A \) is a Banach algebra.

\( A/R \) is the algebra of all sequences \( \sum \alpha_i u_i \) where \( u_i^2 = u_i \), \( \alpha_i \) are complex, and \( \| \sum \alpha_i u_i \| = \left( \sum |\alpha_i|^2 \right)^{1/2} < \infty \). \( A/R \) contains the element \( x = \sum_{i=1}^{\infty} i^{-2} u_i \), since \( \sum i^{-2} = \pi^2/6 \), but there is no element

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\( ^1 \) Numbers in brackets refer to the references cited at the end of the paper.
\[ \sum_{i=1}^{n} e_i \text{ in } A, \text{ for } \sum_{i=1}^{n} e_i \text{ diverges. Therefore there is no subalgebra of } A \text{ isomorphic to } A/R. \]

It can be shown [1] that the radical of \( A \) is one-dimensional. Thus no restriction on the dimension of the radical will suffice. However it will now be shown that it is sufficient for \( A/R \) to be finite-dimensional.

**Theorem 1.** If \( A \) is a Banach algebra, \( R \) its radical, and \( A/R \) is finite-dimensional, then there is a subalgebra \( S \) of \( A \) isomorphic and homeomorphic to \( A/R \). \( A \) is the vector space direct sum \( S + R \).

**Lemma 1.** If \( A \) is a Banach algebra, \( R \) its radical, and \( \{ u_i \} \) a denumerable set of pairwise orthogonal idempotents of \( A/R \), then there exist idempotents \( e_i \) in \( A \) mapping on \( u_i \) via \( A \to A/R \), and the \( e_i \) are pairwise orthogonal.

The proof is by induction. Let \( a_1 \) be an element of \( A \) mapping on the class \( u_1 \). Then \( a_1^2 - a_1 = r_1 \) in \( R \) by hypothesis. For any \( r \) in \( R \) there exists \( (1 + 4r)^{-1/2} = 1 - 2r + 6r^2 - 20r^3 + \cdots \) since \( \| r^n \|^{1/n} \to 0 \) [2] guarantees convergence of this series. Define \( e_1 = (2a_1 - 1)[2(1 + 4r)^{-1/2}]^{-1} + 1/2 = a_1(1 - 2r + 6r^2 - \cdots) + (r - 3r^2 + 10r^3 - \cdots) \). Then \( e_1^2 = e_1 \) and \( e_1 \) maps on \( u_1 \) since \( a_1 \) does. Assume there exist \( e_1, \ldots, e_{t-1} \) such that \( e_i^2 = e_i, e_i e_j = 0 = e_j e_i \) for \( i \neq j \), and \( e_i \to u_i, i = 1, 2, \ldots, t-1 \). Define \( f = \sum_{i=1}^{t-1} e_i \). Then \( f^2 = f, f e_i = e_i f \). Let \( b_i \) be any element such that \( b_i \to u_i \). Define \( a_i = (1-f)b_i(1-f) \). Then \( e_i a_i = a_i e_i = 0, a_i u_i \to 0 \) since \( f b_i \to 0, f b_i \to 0, \) and \( f b_i \to 0 \). Hence \( a_i^2 - a_i = r_i \) in \( R \) and \( e_i r_i = e_i e_i = 0, i = 1, 2, \ldots, t-1 \). Define \( e_i = (2a_i - 1)[2(1 + 4r_i)^{-1/2}]^{-1} + 1/2 \). Then \( e_i^2 = e_i, e_i \to u_i \), and \( e_i e_i = e_i e_i = 0 \) since \( e_i a_i = e_i e_i = 0 \). This completes Lemma 1.

**Lemma 2.** If \( A/R \) contains a ring direct sum \( M_1 \oplus M_2 \oplus \cdots \oplus M_t \) of total matric algebras \( M_i \), then \( A \) contains a ring direct sum of total matric algebras \( S_1 \oplus M_1 \) via \( A \to A/R \).

Consider first a single matric algebra \( M \subset A/R \), where \( M \) is generated over the complexes by \( u_{ij} \), \( u_{ii} \) are pairwise orthogonal idempotents, \( u_{ij} u_{jk} = u_{ik} \), and \( u_{ij} u_{ik} = 0 \) for \( k \neq j \). Since there are a finite number of \( u_{ii} \), by Lemma 1 \( A \) contains idempotents \( e_i \to u_i \) with \( e_i e_{ij} = e_{ij} e_i = 0 \) for \( i \neq j \). Choose an element \( v_{11} \to u_{11} \) and an element \( v_{1j} \to u_{1j} \). Since \( u_{11} u_{11} = u_{11} \) and \( u_{1i} u_{1j} = u_{1j} \), \( v_{1i} \) may be chosen in \( e_{11} e_{11} \); \( v_{1i} \) may be chosen in \( e_{11} e_{1j} \). Then \( v_{1i} v_{1j} \to u_{1j} u_{1j} = u_{11} \). Hence \( v_{1i} v_{1j} = u_{1j} + a_j \) where \( a_j \) is in \( R \cap e_{11} e_{11} \). By [3], \( a_j^* \) exists. \( e_{11} + a_j \) \( (e_{11} + a_j) \) \( = e_{11} + a_j e_{11} + e_j a_j + a_j a_j = e_{11} \) since \( a_j^* = \sum (-a_j) = a_j \) is also in \( e_{11} e_{11} \). Define \( e_{ij} = e_{i1} e_{1j} \). Then \( e_{ij} e_{jk} = e_{ik} \) and \( e_{ij} e_{kh} = 0 \) for \( j \neq k \). Clearly \( e_{ij} \) is
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in \( A \) and \( e_{ij} \rightarrow u_{ij} \). Thus \( A \) contains a total matric algebra \( (e_{ij}) \) isomorphic to \( M \). The sum of the algebras \( S_i \) so constructed for each \( M_i \) is the ring direct sum since the basis elements are constructed from mutually orthogonal idempotents. This completes Lemma 2.

**Proof of Theorem.** \( A/R \) is the direct sum of a finite number of finite-dimensional total matric algebras over the complex numbers. Hence \( A \) contains a subalgebra \( S \cong A/R \). Since the isomorphism \( S \rightarrow A/R \) is continuous, it is a homeomorphism. \( S \) is semi-simple; so \( S \cap R = 0 \). Therefore \( S + R \) is a vector space direct sum.

When \( A/R \) is not finite-dimensional the theorem can still be proved if \( R \) is finite-dimensional and \( A/R \) is a well known type of algebra most generally defined in [4] as follows:

**Definition 2.** The \( B(\infty) \) direct sum of a denumerable number of algebras \( A_i \) is the completion in a specified norm of the algebra of all sequences \( \{a_i\} \) such that \( a_i \) in \( A_i \) are 0 for all but a finite number of \( i \).

**Theorem 2.** If \( A \) is a Banach algebra, the radical \( R \) of \( A \) is finite-dimensional, and \( A/R \) is the \( B(\infty) \) direct sum of finite-dimensional total matric algebras, then \( A \) is a vector space direct sum, \( A = B + C + D \), where \( B \) is finite-dimensional, \( BC = CB = 0 \), every idempotent of \( C \) mapping on an element in the basis of \( A/R \) is orthogonal to \( R \), and \( D \subset R \). When \( A \) is commutative, \( D = 0 \) and \( A \) is a ring direct sum of \( B \) and \( C \).

Let \( n \) be the dimension of \( R \). Then there are at most \( n \) distinct primitive orthogonal idempotents \( e_k \) and \( n \) distinct primitive orthogonal idempotents \( e_s \) of \( A \) for which \( e_k e_s \neq 0 \) and \( r_k e_s \neq 0 \) for any \( r_k \) and \( r_s \) in \( R \). Otherwise

\[
e_{n+1} r_{n+1} = \sum_{k=1}^{n} \alpha_k e_k r_{n+1}, \quad r_{n+1} e_{n+1} = \sum_{l=1}^{n} \beta_l e_l r_{n+1}
\]

for complex \( \alpha_k \) and \( \beta_l \), since any \( n+1 \) elements of \( R \) are linearly dependent. However,

\[
e_{n+1} (e_{n+1} r_{n+1}) = e_{n+1} r_{n+1} = \sum \alpha_k e_{n+1} e_k r_k = 0,
\]

\[(r_{n+1} e_{n+1}) e_{n+1} = r_{n+1} e_{n+1} = \sum \beta_l e_l e_{n+1} = 0.
\]

Hence there are at most \( 2n \) primitive orthogonal idempotents \( e_j \) for which \( e_j R \neq 0 \) or \( Re_j \neq 0 \).

Let \( \{ u_{ij} \} \) be a basis for the matric algebras of \( A/R \). Choose a fixed set of \( e_{ij} \) constructed as in Lemma 2 to map on \( u_{ij} \), and number the set so that \( e_j = e_{jj}, j = 1, \cdots, s \), are all idempotents of the set \( \{e_{ij}\} \) which are not orthogonal to the radical. Define \( e = \sum_{j=1}^{s} e_j, B = eAe, C = (1 - e)A(1 - e), \) and \( D = e(1 - e) + (1 - e)Ae. \) Then \( A = B + C + D \).
is the usual two-sided Peirce decomposition of \( A \). Obviously \( BC = CB = 0 \).

If \( A \) is commutative, \( e(1 - e) = 0 \); so \( D = 0 \). Therefore \( A \) is a ring direct sum, \( A = B \oplus C \).

Note that if \( e_i = e_{ii} \) is an idempotent of \( \{ e_{ij} \} \) which is orthogonal to \( R \) and \( e_k = e_{kk} \) is an idempotent of \( \{ e_{kl} \} \) which maps on \( u_k = u_{kk} \) in the same matric algebra as \( u_{ii} \), then \( e_{kk} \) is also orthogonal to \( R \), since by Lemma 2 there exist \( e_{ik} \) and \( e_{ki} \) such that \( e_{ik}e_{ij}e_{ik} = e_{kk} \). Then \( e_{kk}R = e_{kk}e_{ii}e_{ik}R = 0 \), and \( Re_{kk} = Re_{kk}e_{ii}e_{ik} = 0 \).

Let \( u \) be the image of \( e \) under \( A \to A/R \). Then \( u \) is the sum \( u = I_1 + \cdots + I_n \) where \( I_m \) is the unit element of a matric algebra in \( A/R \). Now \( D \to u(A/R)(1 - u) + (1 - u)(A/R)u \). Since \( u \) commutes with \( A/R \), \( D \to 0 \). Therefore \( DC \subset R \). \( eAe/R \) is finite-dimensional and \( R \) is finite-dimensional. Therefore \( eAe \) is finite-dimensional. All idempotents of \( \{ e_{ij} \} \) not orthogonal to \( R \) are in \( B \); so all idempotents of \( \{ e_{ij} \} \) in \( C \) are orthogonal to \( R \). This completes Theorem 2.

The Principal Theorem of Wedderburn is known for finite-dimensional algebras, so \( B = S_1 + R_1 \). If it can be proved that \( C = S_2 + R_2 \), then it is proved for \( A \); for \( S = S_1 + S_2 \) is a subalgebra, and it follows from \( BC = CB = 0 \) that \( S_1 + S_2 \cong A/R \).

A \( C^* \)-algebra is a Banach algebra with a conjugate linear involution \( x \to x^* \) such that \( (xx^*)^* \) exists for all \( x \) and \( \|xx^*\| = \|x\|^2 \). It is proved in [4] that a completely continuous \( C^* \)-algebra is the \( B(\infty) \) direct sum of finite-dimensional total matric algebras.

**Theorem 3.** If \( A/R \) is a completely continuous \( C^* \)-algebra and \( R \) is finite-dimensional, then \( A \) is a vector space direct sum, \( A = S + R \), of \( R \) and an algebra \( S \) isomorphic and homeomorphic to \( A/R \).

Theorem 2 applies to give \( A = B + C + D \). The remark above implies a continuous isomorphism between \( S_1 \) and \( B/R_1 \). By the closed graph theorem this is a homeomorphism; so it remains to prove the theorem only for the algebra \( C \) in which every idempotent of the set \( \{ e_{ij} \} \) is orthogonal to \( R \). It will thus be assumed that all idempotents in the set \( \{ e_{ij} \} \) are orthogonal to \( R \).

**Lemma 3.** All elements of \( \{ e_{ij} \} \) are orthogonal to \( R \).

Since \( e_{ij} = e_{ii}e_{ij} = e_{ij}e_{jj} \), and it has been assumed that all idempotents are orthogonal to \( R \), it is clear that all \( e_{ij} \) are.

**Lemma 4.** \( \|e_{ij}\| = \|u_{ij}\| = 1 \).

By [5, Theorem 10] and [4] the basis \( \{ u_{ij} \} \) may be chosen so that \( u_{ij}^* = u_{ji} \).
\[ \|u_{i+1}u_{j+1}\| = \|u_{i+1}\| = \|u_{j+1}\| = 1. \]

\[ \|u_{ij}\| = \|u_{ij}\| = \|u_{ij}\| = 1. \]

Hence \( \|u_{ii}\| = 1. \)

By definition,
\[ \inf_{r \in \mathbb{R}} \|e_{ii} + r\| = \|u_{ii}\| = 1. \]

Let \( n \) be the dimension of \( R \). Then \( r^{n+1} = 0. \)

\[ \|(e_{ii} + r)^{n+1}\| = \|e_{ii}^{n+1} + (n + 1)e_{ii}r + \cdots + r^{n+1}\| = \|e_{ii}\| \leq \|e_{ii} + r\|^{n+1}. \]

For any \( \varepsilon > 0 \) there is an \( r \) in \( R \) for which \( \|e_{ii} + r\|^n < 1 + \varepsilon. \) Hence \( \|e_{ii}\| \leq 1. \) Since \( \|e_{ii}\| \leq \|e_{ii}\|^2, \|e_{ii}\| = 1. \) Now
\[ \inf_{r \in \mathbb{R}} \|e_{ii} + r\| = \|u_{ii}\| = 1, \]
\[ \inf_{r \in \mathbb{R}} \|e_{ij} + r\| = \|u_{ij}\| = 1, \]
\[ e_{ii} = (e_{ii} + r)e_{ii}, \]
\[ \|e_{ii}\| \leq \|e_{ii} + r\| e_{ii} = \|e_{ii} + r\|, \]
\[ e_{ij} = e_{ii}(e_{ij} + r), \]
\[ \|e_{ij}\| \leq \|e_{ii}\| e_{ij} + r\| = \|e_{ij} + r\|. \]

This shows \( \|e_{ii}\| \leq 1 \) and \( \|e_{ij}\| \leq 1. \) The mapping \( e_{ij} \to u_{ij} \) depresses the norm.
\[ \|e_{ii}\| \leq \|e_{ii}\| e_{ij} \| = 1. \]

Therefore \( \|e_{ij}\| = 1. \)

**Proof of theorem.** \( A/R \) is the \( B(\infty) \) direct sum of finite-dimensional total matric algebras \( M_i. \) By Lemma 4, \( A \) contains a subalgebra \( S_i \) equivalent to \( M_i. \) It will be shown that the map of any finite sum \( \sum_{i=1}^{n} N_i, N_i \) in \( S_i, \) into \( A/R \) is an isometry. Suppose \( N_i \to N_i \) in \( M_i, \) and \( I_i \) is the identity matrix of \( S_i. \) Since \( A/R \) is a \( C^* \)-algebra, \( (I_iI_i^*)^* = I_1 = I_1I_1^*, \) \( \|I_1\| = \|I_1I_1^*\| = \|I_1\|^2; \) so \( \|I_1\| = 1. \) Furthermore
\[ \|I_1 + \cdots + I_4\| = \|(I_1 + \cdots + I_i)(I_i^* + \cdots + I_4^*)\|^2, \]
\[ \|I_1 + \cdots + I_4\| = \|I_1 + \cdots + I_4\|^2 = 1. \]

Define \( I = I_1 + \cdots + I_4. \) Then
\[ \inf_{r \in R} \|I + r\| = \|I\| = 1. \]
\[ \|(I + r)^{n+1}\| = \|I\| \leq \|I + r\|^{n+1}. \]

Hence \( \|I\| = 1, \) and similarly
\[
\inf_{r \in R} \left\| \sum_{i=1}^{t} N_i + r \right\| = \left\| \sum N_i \right\|
\]
\[
I(\sum N_i + r) = \sum N_i, \quad \left\| \sum N_i \right\| \leq \left\| I \right\| \left\| \sum N_i + r \right\| = \left\| \sum N_i + r \right\|, \quad \left\| \sum N_i \right\| \leq \left\| I \right\|
\]

Since the mapping \( A \to A/R \) depresses norms,
\[
\left\| \sum N_i \right\| = \left\| \sum N_i \right\|
\]

This shows that the mapping of any finite sum \( \sum_{i=1}^{t} N_i \) into \( A/R \) is an isometry. Let \( S \) be the \( B(\infty) \) direct sum of the subalgebras \( S_i \) of \( A \). Since \( A \) is complete, \( S \subseteq A \). A dense subset of \( S \) maps isometrically and isomorphically onto a dense subset of \( A/R \); therefore \( S \) is isomorphic and isometric to \( A/R \). This proves Theorem 3.

The theorem will now be proved for an algebra in which the mapping \( A \to A/R \) depresses the norm as little as possible.

**Definition 3.** An \( l_1 \) algebra is the commutative Banach algebra of all sums \( \sum \alpha_i u_i \), where \( \alpha_i \) are complex, \( u_i \) are a denumerable number of primitive orthogonal idempotents, and \( \left\| \sum \alpha_i u_i \right\| = \sum |\alpha_i| < \infty \).

**Theorem 4.** If \( A/R \) is an \( l_1 \) algebra and \( R \) is finite-dimensional, then \( A = S + R \) where \( S \) is a subalgebra of \( A \) isomorphic and homeomorphic to \( A/R \).

As in Theorem 3 it is sufficient to consider an algebra \( A \) in which each idempotent \( e_i \) is orthogonal to \( R \).

There exist pairwise orthogonal idempotents \( e_i \to u_i \) by Lemma 1. The proof of Lemma 4 shows \( \|e_i\| = 1 \). For any \( x = \sum \alpha_i e_i \) in \( A \),
\[
\|x\| \leq \left\| \sum \alpha_i e_i \right\| \leq \sum |\alpha_i| \|e_i\| = \sum |\alpha_i| = \left\| \sum \alpha_i u_i \right\|
\]
and the mapping \( A \to A/R \) decreases norms. Hence \( \|x\| = \sum |\alpha_i| \), that is, the mapping is an isometry on the completion \( S \) of the subalgebra of \( A \) generated by the \( e_i \). Therefore \( S \) is isometric and isomorphic to \( A/R \) and \( A = S + R \). This completes the proof.

In all the previous theorems the completion of the algebra generated by elements mapping on basis elements of \( A/R \) is disjoint from the radical. The following theorem shows this property is the essential one.

**Theorem 5.** Suppose \( A \) is a Banach algebra, that the radical \( R \) is finite-dimensional, that \( A/R \) is the \( B(\infty) \) sum of finite-dimensional total matric algebras, that \( S \) is the \( B(\infty) \) sum in \( A \) of the matric algebras isomorphic to those of \( A/R \), and that \( S \cap R = 0 \). Then \( S \) is isomorphic and homeomorphic to \( A/R \), and \( A \) is the vector space direct sum \( S + R \).
$S$ is complete and $R$ is complete since the radical of a Banach algebra is closed. $R$ is finite-dimensional so $S + R$ is complete. Also $(S + R)/R$ is complete; hence $A \to A/R$ maps $S + R$ onto $A/R$. $S \cap R = 0$ implies $(S + R)/R = S$. Therefore $S \cong A/R$. The mapping $S \to A/R$ is 1-1 and continuous. By the closed graph theorem, $S$ is homeomorphic to $A/R$. Suppose $a$ in $A$ maps on $[a]$ in $A/R$. Then there is an $s$ in $S$ which maps on $[a]$. Thus $a - s = r$ in $R$. Every $a = s + r$. Since $S$ is semi-simple, $A = S + R$.

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