A CHARACTERIZATION OF SIMPLY CONNECTED CLOSED ARCWISE CONVEX SETS

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Let S be a set of points in the Euclidean plane E_2. It is our purpose to establish a necessary and sufficient condition that a simply connected[1] closed set S be arcwise convex. In order to do this precisely, the following notations and definitions are used.

NOTATION. The line determined by two distinct points x and y in E_2 is denoted by $L(x, y)$. We designate the open line segment joining x and y by xy, and the corresponding closed segment by $[xy]$. The two closed half-planes having $L(x, y)$ as a common boundary are designated by $R_1(x, y)$ and $R_2(x, y)$. The boundary of a set K is represented by $B(K)$, and $H(K)$ denotes the convex hull of K. The complement of S is denoted by $C(S)$.

DEFINITION 1. *A set $S \subset E_2$ is said to be unilaterally connected if, for each pair of distinct points x and y in S, there exists a continuum*[2] *$S_1 \subset S$ which contains x and y, and which lies in one of the closed half-planes determined by $L(x, y)$.*

DEFINITION 2. *A set $S \subset E_2$ is said to be arcwise convex if each pair of points in S can be joined by a convex arc lying in S. (A convex arc is one which is contained in the boundary of its convex hull.)*

In a previous paper [1][3] the author studied the complements of both arcwise convex sets and unilaterally connected sets. The theorem below establishes another intimate connection between these two concepts.

I am indebted to the referee for the following lemma which simplifies the proof of the theorem.

LEMMA. *In order that a simply connected, connected, closed set $S \subset E_2$ be unilaterally connected, it is necessary that for each line L, all of the bounded components of $C(S) - L \cdot C(S)$ lie on the same side of L.*

PROOF. Suppose L is a straight line for which a bounded component D of $C(S) - L \cdot C(S)$ exists. Let $[xy]$ be the minimal closed interval containing $L \cdot B(D)$. Let T be a continuum in S which con-

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[1] A set $S \subset E_2$ is simply connected if each component of its complement is unbounded.

[2] A continuum in E_2 is a bounded, closed, connected set.

[3] Number in brackets refers to the reference at the end of the paper.

tains $x+y$, and which lies in a closed half-plane, denoted by $R_1(x, y)$, determined by L. There exists a circular circumference Q which encloses $T+[xy]+B(D)$. Since $x+y\subset T$, no two arcs which intersect Q but not T can abut $[xy]$ from opposite sides. Let A be an arc in $C(S)$ irreducible from $[xy]$ to Q. (By definition, A contains no proper subarc containing points of $[xy]$ and points of Q.) Then A abuts on $[xy]$, and it also contains an arc in $R_2(x, y)$ abutting on $[xy]$. Moreover, it is clear that $A\cdot D=0$.

Suppose that $D\subset R_2(x, y)$. Let Q_x and Q_y denote closed circular disks centered on x and y respectively, such that $(Q_x+Q_y)\cdot(A+Q)=0$. There exists an arc $E\subset D+B(Q_x)+B(Q_y)$, having only its end points, w and z, in $L\cdot xy$, such that $A\cdot xy$ is between w and z on L. Then $E+wz$ is a simple closed curve enclosed by Q and lying in $R_2(x, y)$. Since A abuts on $[xy]$ via $R_2(x, y)$, the above implies that $A-A\cdot xy$ lies within the region bounded by $E+wz$. This is a contradiction, so that we have $D\subset R_1(x, y)$.

If U is any other bounded component of $C(S)-L\cdot C(S)$, let $[pq]$ denote the minimal closed interval of L containing $L\cdot B(U)$. Each pair of the four points x, y, p, q (whether distinct or not) is contained in a continuum in S lying in $R_1(x, y)$ or in $R_2(x, y)$. From this fact it follows readily that there exists a continuum $T'\subset S$ which contains $x+y+p+q$, and which lies in $R_1(x, y)$ or in $R_2(x, y)$. From the above paragraph we must have $T'\subset R_1(x, y)$ since $D\subset R_1(x, y)$. Hence, we must also have $U\subset R_1(x, y)$. This completes the proof.

THEOREM. *A necessary and sufficient condition that a simply connected closed set $S\subset E_2$ be arcwise convex is that it be unilaterally connected.*

PROOF. It is the sufficiency which requires proof, since the necessity is obvious. Choose $x\in S$, $y\in S$. If $xy\subset S$, then x and y can be joined by a convex arc in S. Hence, suppose $xy\not\subset S$. By hypothesis, there exists a continuum $S_1\subset S$ containing x and y and lying in $R_1(x, y)$ or in $R_2(x, y)$. Suppose $S_1\subset R_1(x, y)$. Choose any point $\alpha\in xy\cdot C(S)$. Define $K(\alpha)$ to be that component of $C(S)\cdot R_1(x, y)$ which contains α. Since S_1 is a bounded closed connected set in $R_1(x, y)$, and since we can establish an order $x<\alpha<y$ on $L(x, y)$, we have $K(\alpha)\subset H(S_1)$. Hence the set sum $\sum K(\alpha)$ (α ranges over $C(S)\cdot xy$) is bounded. Define the set sum T to be

$$T \equiv x + y + \overline{\sum K(\alpha)} \qquad (\alpha \text{ ranges over } C(S)\cdot xy).$$

It follows with the help of the preceding lemma that $C\equiv B(H(T))-xy$ is a convex arc lying in S. This proves the theorem.

The above characterization does not apply to sets which are not simply connected. For instance, the set S consisting of the circumference of a circle C plus a single outward normal to C (segment or half-line) is unilaterally connected but not arcwise convex. A nontrivial characterization of non-simply connected arcwise convex sets appears to be difficult to determine.

Reference

1. F. A. Valentine, *Arcwise convex sets*, Proceedings of the American Mathematical Society vol. 2 (1951) pp. 159–165.

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