A CHARACTERIZATION OF SIMPLY CONNECTED CLOSED ARCWISE CONVEX SETS

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Let \( S \) be a set of points in the Euclidean plane \( E_2 \). It is our purpose to establish a necessary and sufficient condition that a simply connected\(^1\) closed set \( S \) be arcwise convex. In order to do this precisely, the following notations and definitions are used.

Notation. The line determined by two distinct points \( x \) and \( y \) in \( E_2 \) is denoted by \( L(x, y) \). We designate the open line segment joining \( x \) and \( y \) by \( xy \), and the corresponding closed segment by \([xy]\)\. The two closed half-planes having \( L(x, y) \) as a common boundary are designated by \( R_1(x, y) \) and \( R_2(x, y) \). The boundary of a set \( K \) is represented by \( B(K) \), and \( H(K) \) denotes the convex hull of \( K \). The complement of \( S \) is denoted by \( C(S) \).

Definition 1. A set \( S \subseteq E_2 \) is said to be unilaterally connected if, for each pair of distinct points \( x \) and \( y \) in \( S \), there exists a continuum\(^2\) \( S_1 \subseteq S \) which contains \( x \) and \( y \), and which lies in one of the closed half-planes determined by \( L(x, y) \).

Definition 2. A set \( S \subseteq E_2 \) is said to be arcwise convex if each pair of points in \( S \) can be joined by a convex arc lying in \( S \). (A convex arc is one which is contained in the boundary of its convex hull.)

In a previous paper \[1\]\(^3\) the author studied the complements of both arcwise convex sets and unilaterally connected sets. The theorem below establishes another intimate connection between these two concepts.

I am indebted to the referee for the following lemma which simplifies the proof of the theorem.

Lemma. In order that a simply connected, connected, closed set \( S \subseteq E_2 \) be unilaterally connected, it is necessary that for each line \( L \), all of the bounded components of \( C(S) - L \cdot C(S) \) lie on the same side of \( L \).

Proof. Suppose \( L \) is a straight line for which a bounded component \( D \) of \( C(S) - L \cdot C(S) \) exists. Let \([xy]\) be the minimal closed interval containing \( L \cdot B(D) \). Let \( T \) be a continuum in \( S \) which con-

\(^1\) A set \( S \subseteq E_2 \) is simply connected if each component of its complement is unbounded.
\(^2\) A continuum in \( E_2 \) is a bounded, closed, connected set.
\(^3\) Number in brackets refers to the reference at the end of the paper.

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tains \(x+y\), and which lies in a closed half-plane, denoted by \(R_1(x, y)\),
determined by \(L\). There exists a circular circumference \(Q\) which en-
closes \(T + [xy] + B(D)\). Since \(x+y \subseteq T\), no two arcs which intersect \(Q\)
but not \(T\) can abut \([xy]\) from opposite sides. Let \(A\) be an arc in \(C(S)\)
irreducible from \([xy]\) to \(Q\). (By definition, \(A\) contains no proper
subarc containing points of \([xy]\) and points of \(Q\).) Then \(A\) abuts on
\([xy]\), and it also contains an arc in \(R_2(x, y)\) abutting on \([xy]\). Moreover,
it is clear that \(A \cdot D = 0\).

Suppose that \(D \subseteq R_2(x, y)\). Let \(Q_x\) and \(Q_y\) denote closed circular
disks centered on \(x\) and \(y\) respectively, such that \((Q_x + Q_y) \cap (A + Q) = 0\).
There exists an arc \(E \subseteq D + B(Q_x) + B(Q_y)\), having only its end
points, \(w\) and \(z\), in \(L \cdot xy\), such that \(A \cdot xy\) is between \(w\) and \(z\) on \(L\).
Then \(E + wz\) is a simple closed curve enclosed by \(Q\) and lying in
\(R_2(x, y)\). Since \(A\) abuts on \([xy]\) via \(R_1(x, y)\), the above implies that
\(A - A \cdot xy\) lies within the region bounded by \(E + wz\). This is a con-
tradiction, so that we have \(D \subseteq R_1(x, y)\).

If \(U\) is any other bounded component of \(C(S) - L \cdot C(S)\), let \([pq]\)
denote the minimal closed interval of \(L\) containing \(L \cdot B(U)\). Each
pair of the four points \(x, y, p, q\) (whether distinct or not) is contained
in a continuum in \(S\) lying in \(R_1(x, y)\) or in \(R_2(x, y)\). From this fact it
follows readily that there exists a continuum \(T' \subseteq S\) which contains
\(x+y+p+q\), and which lies in \(R_1(x, y)\) or in \(R_2(x, y)\). From the above
paragraph we must have \(T' \subseteq R_1(x, y)\) since \(D \subseteq R_1(x, y)\). Hence, we
must also have \(U \subseteq R_1(x, y)\). This completes the proof.

Theorem. A necessary and sufficient condition that a simply con-
nected closed set \(S \subseteq \mathbb{E}_2\) be arcwise convex is that it be unilaterally con-
nected.

Proof. It is the sufficiency which requires proof, since the necessity
is obvious. Choose \(x \in S, y \in S\). If \(xy \subseteq S\), then \(x\) and \(y\) can be joined
by a convex arc in \(S\). Hence, suppose \(xy \notin S\). By hypothesis, there
exists a continuum \(S_1 \subseteq S\) containing \(x\) and \(y\) and lying in \(R_1(x, y)\) or
in \(R_2(x, y)\). Suppose \(S_1 \subseteq R_1(x, y)\). Choose any point \(\alpha \in xy \cdot C(S)\). De-
fine \(K(\alpha)\) to be that component of \(C(S) \cdot R_1(x, y)\) which contains \(\alpha\).
Since \(S_1\) is a bounded closed connected set in \(R_1(x, y)\), and since we
can establish an order \(x < \alpha < y\) on \(L(x, y)\), we have \(K(\alpha) \subseteq H(S_1)\).
Hence the set sum \(\sum K(\alpha) (\alpha \text{ ranges over } C(S) \cdot xy)\) is bounded. De-
finite the set sum \(T\) to be

\[
T = x + y + \sum K(\alpha) \quad (\alpha \text{ ranges over } C(S) \cdot xy).
\]

It follows with the help of the preceding lemma that \(C = B(H(T)) - xy\)
is a convex arc lying in \(S\). This proves the theorem.
The above characterization does not apply to sets which are not simply connected. For instance, the set $S$ consisting of the circumference of a circle $C$ plus a single outward normal to $C$ (segment or half-line) is unilaterally connected but not arcwise convex. A nontrivial characterization of non-simply connected arcwise convex sets appears to be difficult to determine.

Reference


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