A CHARACTERIZATION OF SIMPLY CONNECTED
CLOSED ARCWISE CONVEX SETS

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Let $S$ be a set of points in the Euclidean plane $E_2$. It is our purpose to establish a necessary and sufficient condition that a simply connected\(^1\) closed set $S$ be arcwise convex. In order to do this precisely, the following notations and definitions are used.

**Notation.** The line determined by two distinct points $x$ and $y$ in $E_2$ is denoted by $L(x, y)$. We designate the open line segment joining $x$ and $y$ by $xy$, and the corresponding closed segment by $[xy]$. The two closed half-planes having $L(x, y)$ as a common boundary are designated by $R_1(x, y)$ and $R_2(x, y)$. The boundary of a set $K$ is represented by $B(K)$, and $H(K)$ denotes the convex hull of $K$. The complement of $S$ is denoted by $C(S)$.

**Definition 1.** A set $S \subseteq E_2$ is said to be unilaterally connected if, for each pair of distinct points $x$ and $y$ in $S$, there exists a continuum\(^2\) $S \subseteq S$ which contains $x$ and $y$, and which lies in one of the closed half-planes determined by $L(x, y)$.

**Definition 2.** A set $S \subseteq E_2$ is said to be arcwise convex if each pair of points in $S$ can be joined by a convex arc lying in $S$. (A convex arc is one which is contained in the boundary of its convex hull.)

In a previous paper \([1]\)\(^3\) the author studied the complements of both arcwise convex sets and unilaterally connected sets. The theorem below establishes another intimate connection between these two concepts.

I am indebted to the referee for the following lemma which simplifies the proof of the theorem.

**Lemma.** In order that a simply connected, connected, closed set $S \subseteq E_2$ be unilaterally connected, it is necessary that for each line $L$, all of the bounded components of $C(S) - L \cdot C(S)$ lie on the same side of $L$.

**Proof.** Suppose $L$ is a straight line for which a bounded component $D$ of $C(S) - L \cdot C(S)$ exists. Let $[xy]$ be the minimal closed interval containing $L \cdot B(D)$. Let $T$ be a continuum in $S$ which con-

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\[^1\] A set $S \subseteq E_2$ is simply connected if each component of its complement is unbounded.

\[^2\] A continuum in $E_2$ is a bounded, closed, connected set.

\[^3\] Number in brackets refers to the reference at the end of the paper.
SIMPLY CONNECTED CLOSED ARCWISE CONVEX SETS

...contains \( x + y \), and which lies in a closed half-plane, denoted by \( R_1(x, y) \), determined by \( L \). There exists a circular circumference \( Q \) which encloses \( T + [xy] + B(D) \). Since \( x + y \subset T \), no two arcs which intersect \( Q \) but not \( T \) can abut \([xy]\) from opposite sides. Let \( A \) be an arc in \( C(S) \) irreducible from \([xy]\) to \( Q \). (By definition, \( A \) contains no proper subarc containing points of \([xy]\) and points of \( Q \).) Then \( A \) abuts on \([xy]\), and it also contains an arc in \( R_2(x, y) \) abutting on \([xy]\). Moreover, it is clear that \( A \cdot D = 0 \).

Suppose that \( D \subset R_2(x, y) \). Let \( Q_x \) and \( Q_y \) denote closed circular disks centered on \( x \) and \( y \) respectively, such that \((Q_x + Q_y) \cap (A + Q) = 0\). There exists an arc \( E \subset D + B(Q_x) + B(Q_y) \), having only its end points, \( w \) and \( z \), in \( L \cdot xy \), such that \( A \cdot xy \) is between \( w \) and \( z \) on \( L \). Then \( E + wz \) is a simple closed curve enclosed by \( Q \) and lying in \( R_2(x, y) \). Since \( A \) abuts on \([xy]\) via \( R_1(x, y) \), the above implies that \( A - A \cdot xy \) lies within the region bounded by \( E + wz \). This is a contradiction, so that we have \( D \subset R_1(x, y) \).

If \( U \) is any other bounded component of \( C(S) - L \cdot C(S) \), let \([pq]\) denote the minimal closed interval of \( L \) containing \( L \cdot B(U) \). Each pair of the four points \( x, y, p, q \) (whether distinct or not) is contained in a continuum in \( S \) lying in \( R_1(x, y) \) or in \( R_2(x, y) \). From this fact it follows readily that there exists a continuum \( T' \subset S \) which contains \( x + y + p + q \), and which lies in \( R_1(x, y) \) or in \( R_2(x, y) \). From the above paragraph we must have \( T' \subset R_1(x, y) \) since \( D \subset R_1(x, y) \). Hence, we must also have \( U \subset R_1(x, y) \). This completes the proof.

**Theorem.** A necessary and sufficient condition that a simply connected closed set \( S \subset E_2 \) be arcwise convex is that it be unilaterally connected.

**Proof.** It is the sufficiency which requires proof, since the necessity is obvious. Choose \( x \in S, y \in S \). If \( xy \subset S \), then \( x \) and \( y \) can be joined by a convex arc in \( S \). Hence, suppose \( xy \subset S \). By hypothesis, there exists a continuum \( S_1 \subset S \) containing \( x \) and \( y \) and lying in \( R_1(x, y) \) or in \( R_2(x, y) \). Suppose \( S_1 \subset R_1(x, y) \). Choose any point \( \alpha \in xy \cdot C(S) \). Define \( K(\alpha) \) to be that component of \( C(S) \cdot R_1(x, y) \) which contains \( \alpha \). Since \( S_1 \) is a bounded closed connected set in \( R_1(x, y) \), and since we can establish an order \( x < \alpha < y \) on \( L(x, y) \), we have \( K(\alpha) \subset H(S_1) \).

Hence the set sum \( \sum K(\alpha) \) (\( \alpha \) ranges over \( C(S) \cdot xy \)) is bounded. Define the set sum \( T \) to be

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T = x + y + \sum K(\alpha) \quad (\alpha \text{ ranges over } C(S) \cdot xy).
\]

It follows with the help of the preceding lemma that \( C = B(H(T)) - xy \) is a convex arc lying in \( S \). This proves the theorem.
The above characterization does not apply to sets which are not simply connected. For instance, the set $S$ consisting of the circumference of a circle $C$ plus a single outward normal to $C$ (segment or half-line) is unilaterally connected but not arcwise convex. A nontrivial characterization of non-simply connected arcwise convex sets appears to be difficult to determine.

**Reference**