NOTE ON THE LOCATION OF THE CRITICAL POINTS
OF A REAL RATIONAL FUNCTION

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The object of this note is to study the critical points of a real rational function \( p(z) \) of degree \(-2m\) \((>0)\) with \(-2m\) poles in the closed interior of the unit circle \( C: |z|=1 \), with a real zero of multiplicity \( k \) \((>0)\) in the point \( x_1 \) interior to \( C \), and with a zero of multiplicity \(-2m-k\) \((>0)\) at infinity. The corresponding problems for the case \( |x_1|>1 \), and for the case that \( p(z) \) is of degree \( k \) with infinity no longer a zero but a pole of \( p(z) \) of multiplicity \( k+2m \) \((>0)\) with no restrictions on \( |x_1| \), have already been treated elsewhere;\(^1\) we retain the notation and terminology of that previous treatment.

We shall prove

**Lemma 5.** Let \( C: |z|=1 \) be the unit circle, let \( \lambda \) \((>1)\) be constant, and let \( A: z=x_1 \) be a real point interior to \( C \). If \( P \) is a variable nonreal point, we denote by \( Q \) the intersection other than \( P \) of the line \( AP \) with the circle through \(-1, +1, \text{ and } P\). Then the locus of points \( P: (x, y) \) such that we have \( QP/AP=\lambda \) consists of the nonreal points of

\[ (9) \quad (\lambda - 1) \left[ (x - x_1)^2 + y^2 \right] + x_1 - 1 = 0. \]

If \( P \) is the point \((x, y)\), then the circle through \(-1, +1, \text{ and } P\) is

\[ (10) \quad X^2 + \left[ Y - \left( x^2 + y^2 - 1 \right)/2y \right]^2 = 1 + \left( x^2 + y^2 - 1 \right)^2/4y^2, \]

where the running coordinates are \( X \) and \( Y \). The point \( Q \) has the coordinates \((\lambda x_1+x-\lambda x, y-\lambda y)\), and a necessary and sufficient condition that \( Q \) lie on the circle \((10)\) is precisely \((9)\). Equation \((9)\) represents a proper circle.

**Lemma 6.** Let \( C: |z|=1 \) be the unit circle, and \( A: z=x_1 \) a real point interior to \( C \). If the point \( P: (x, y) \) lies exterior to the circle \((9)\), then for every point \( Q' \) collinear with \( A \) and \( P \), separated by \( A \) from \( P \), and lying interior to the circle \((10)\), we have \( Q'P/AP<\lambda \).

It is to be noted that \( A \) is the center of the circle \((9)\). When \( P \) moves on \( AP \) monotonically away from \( A \), the point \( Q \) of Lemma 5 moves monotonically toward \( A \). Consequently the ratio \( QA/AP \)

\(^1\) J. L. Walsh, *The location of critical points*, Amer. Math. Soc. Colloquium Publications, vol. 34, §5.8.3. All references in the present note are to that book.
Theorem 8. Let \( p(z) \) be a rational function of degree \(-2m\) all of whose poles lie in the closed interior of the unit circle \( C \), which has a zero of multiplicity \( k \) (\( > 0 \)) in the real point \( x_1 \) interior to \( C \), and which has a zero of multiplicity \(-2m-k\) (\( > 0 \)) at infinity. Then all nonreal finite critical points of \( p(z) \) exterior to \( C \) lie in the closed interior of the circle

\[
(k + 2m)[(x - x_1)^2 + y^2] + k(1 - x_1^2) = 0.
\]

Let \( P \) be a nonreal critical point of \( p(z) \) exterior to \( C \), hence a position of equilibrium in the usual field of force. The particles at the poles of \( p(z) \) total \( 2m \) in mass, are symmetric in the axis of reals, and lie in the closed interior of \( C \); hence, by the method of proof of Lemma 2, the force these particles exert at \( P \) is equal to the force exerted at \( P \) by a single particle of mass \( 2m \) located at some point \( Q' \) of \( R \) (notation of Lemma 2). Then \( P \) is a position of equilibrium in the field due to this particle at \( Q' \), and to a particle of mass \( k \) in the point \( A : z = x_1 \). Consequently \( Q' \) lies in \( R_0 \) (notation of Lemma 2), \( P \) is collinear with \( A \) and \( Q' \), and we have \( Q'P/AP = -2m/k \). It follows from Lemma 6 with \( X = -2m/k \) that \( P \) lies on or within the circle (11). Theorem 8 is established.

From a general result on circular regions (§4.2.4, Theorem 4), it follows that under the hypothesis of Theorem 8 all real critical points of \( p(z) \) lie in the interval \( S \):

\[
(-2mx_1 - k)/(-2m - k) \leq s \leq (-2mx_1 + k)/(-2m - k).
\]

In order to study the actual locus of critical points in Theorem 8, we consider a somewhat more general situation:

Theorem 9. Let a region \( R \) symmetric in the axis of reals be the locus of \( q \) (\( > 4 \)) poles of a real rational function \( p(z) \). Then each point of \( R \) belongs to the locus of critical points of \( p(z) \).

As in the proof of §4.2.2, Theorem 1, we find it convenient to consider \( R(z) = 1/p(z) \), and shall prove that each (interior) point of \( R \) can be a critical point not a multiple zero of \( R(z) \). This conclusion is valid for a real point \( \alpha_0 \) of \( R \), for let \( \alpha_0 \) lie interior to \( R \) on the axis of reals; precisely the method of §4.2.2 then shows that \( \alpha_0 \) can be a critical point not a multiple zero of \( R(z) \).

Let now \( \alpha_0 \) be a nonreal point of \( R \); we set

\[
R(z) = (z - \alpha_0)(z - \bar{\alpha}_0)(s - \alpha)(s - \bar{\alpha})R_1(z),
\]
where $\alpha$ is allowed to vary in the neighborhood of the fixed point $\alpha_0$, but where the zeros and poles of $R_1(z)$ are fixed and remote from $\alpha_0$. The critical points of $R(z)$ are the zeros of $R'(z) = (z - \alpha_0)(z - \alpha)R_1(z) + (z - \alpha_0)(z - \alpha_0)R_1(z) + (z - \alpha_0)(z - \alpha)(z - \alpha)R_1(z) + (z - \alpha_0)(z - \alpha_0)(z - \alpha)R_1(z) = 0$.

This equation defines $z$ as an implicit function of $\alpha$, and the equation is satisfied for $z = \alpha = \alpha_0$. Even though $z$ is not an analytic function of $\alpha$, we have by differentiation for the values $z = \alpha = \alpha_0$

$$\frac{\partial R'(z)}{\partial z} = 2(\alpha_0 - \bar{\alpha}_0)^2 R_1(\alpha_0),$$

$$\frac{\partial R'(z)}{\partial \alpha} = -(\alpha_0 - \bar{\alpha}_0)^2 R_1(\alpha_0).$$

We interpret $R'(z) = 0$ as two real equations in four real variables, the real and pure imaginary parts of $z$ and $\alpha$; the jacobian for the values $z = \alpha = \alpha_0$ has a value which is different from zero. It follows from the implicit function theorem for real variables that the equation $R'(z) = 0$ defines $z$ as a function of $\alpha$, and when $\alpha$ varies throughout a neighborhood of $\alpha_0$ then $z$ also varies throughout a neighborhood of $\alpha_0$; it is readily shown also from the proof of the implicit function theorem based on successive approximations that if $\alpha_0$ is now allowed to vary over a small neighborhood, then a neighborhood of $\alpha_0$ of fixed size as a locus of $z$ which contains uniformly a circle of constant positive radius whose center is the variable $\alpha_0$. The proof of Theorem 9 can now be completed as was the proof of §4.2.2, Theorem 1.

We return now to Theorem 8. If we replace the problem represented by Theorem 8 by the more general problem of finding the critical points of a function of the form $(0 < k < -2m)$

$$\hat{p}(z) = (z - x_1)^k/(z - \alpha_1)^m(z - \bar{\alpha}_1)^m(z - \alpha_2)^m(z - \bar{\alpha}_2)^m \cdots (z - \alpha_n)^m(z - \bar{\alpha}_n)^m,$$

(12)

where we have $|x_1| < 1$, $|\alpha_j| \leq 1$, $m_j > 0$, $m_1 + m_2 + \cdots + m_n = -m$, and where $k$, $m$, $x_1$ are given but $k$, $m_1$, $m_2$, $\cdots$, $m_n$, $m$ need no longer be integral, then the interior of $C$ plus the closed interior of (11) plus $S$ plus the point at infinity is the precise locus of critical points of $\hat{p}(z)$. Any point $z$ interior to $C$ can be a critical point of $\hat{p}(z)$, as follows from Theorem 9. Any point of $S$ can be a critical point of $\hat{p}(z)$; compare §4.2.4, Theorem 4. Any nonreal point $P$ in the closed interior of (9) not interior to $C$ can be a critical point of $\hat{p}(z)$, for if $P$ is given there.
exists a point $Q'$ on the line $PA$ extended, which is contained in the closed region $R_0$ (notation of Lemma 2) and with $Q'P/AP = -2m/k$; it follows from the proof of Lemma 2 that a suitable choice of negative particles in $C$ of total mass $2m$ is equivalent (so far as concerns the force at $P$) to a $(2m)$-fold negative particle at $Q'$; thus $P$ is a critical point of a suitably chosen $p(z)$ of type (12). Every real point interior to (11) lies in $C$ or $S$, and the conclusion of Theorem 8 persists, so the locus of critical points of (12) is as stated.

Under the original conditions of Theorem 8, with $x_1$ given, every real point interior to $C$ belongs to the locus of critical points if we have $-m = 1$, and every point interior to $C$ belongs to the locus if we have $-m > 1$.

If the hypothesis of Theorem 8 is modified slightly, the conclusion requires large modification:

**Theorem 10.** Let $p(z)$ be a rational function of form (12) where we have $|x_1| < 1$, $|\alpha_j| \leq 1$, $m_j > 0$, $m_1 + m_2 + \cdots + m_n = -m$, and where $m$, $x_1$ are given, but $k$, $m_1$, $m_2$, $\cdots$, $m_n$, $m$ need not be integral. Then the locus of critical points of $p(z)$ for all possible choices of $k$, $m_1$, $m_2$, $\cdots$, $m_n$, $\alpha$, consists of the extended plane.

The proof of Theorem 10 is similar to the previous proof of the locus property of (12), and is left to the reader.